Object Formation and Conceptual Resolution: Stable Quotients, Complete Refinement Lattices, and Functorial Observation

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Mathematical theories ordinarily begin with objects already identified, but they rarely formalize the observational distinctions through which those objects become available. This paper develops the first algebraic foundation for object formation by treating an observation policy as an equivalence relation on a state space. Each policy produces a quotient whose classes are the objects distinguishable at that conceptual resolution. When the equivalence relation is preserved by admissible transformations, those transformations descend coherently to the quotient, yielding stable object identity. The full family of observation policies forms a complete lattice under refinement and, equivalently, a thin category whose arrows represent the lawful forgetting of distinctions. Maps between state spaces pull observation policies backward, making conceptual resolution contravariantly functorial. State spaces equipped with observation policies then assemble into a global category whose morphisms preserve observational identity and induce maps between quotient object spaces. Concept birth becomes strict refinement, object fusion becomes coarsening, and destructive abstraction becomes failure of transformation invariance. These results provide the first rigorous categorical structure of the discovery calculus proposed in earlier work on the mathematics of concept formation.

Keywords: concept formation, object formation, stable quotients, observation policies, equivalence relations, conceptual resolution, complete lattices, refinement categories, contravariant functors, quotient spaces, discovery calculus, categorical observation, policy-indexed identity, mathematical cognition, artificial intelligence.

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1. Introduction: From Concept Formation to Formal Resolution

Mathematics usually begins after its objects have already been chosen. A set is given, a space is specified, a function is defined, or a category is constructed. Once those objects exist, mathematical reasoning proceeds through operations, transformations, and proofs. What remains less frequently formalized is the prior step through which a state of affairs becomes partitioned into distinct objects in the first place. Before a theory can study objects, it must possess a policy for deciding which differences matter, which variations can be ignored, and when two states should count as the same.

The earlier paper, The Mathematics of Concept Formation: Discovery Calculus, Explanation Geometry, Theory Dynamics, and the Formal Birth of New Objects, proposed that concept formation should itself become a mathematical subject. It argued that observation, explanation, compression, error correction, theory change, and object birth share recurring structures that can be analyzed formally. That paper established a broad research program. The present work narrows the program to its first exact algebraic construction.

The central idea is simple. An observation policy determines which states can be distinguished. States that the policy cannot distinguish are treated as equivalent. The corresponding equivalence classes become the effective objects visible at that resolution. If the equivalence relation is preserved by the transformations regarded as admissible, those transformations act consistently on the quotient classes. The quotient then supports stable identity, lawful transformation, and representative-independent reasoning.

This construction provides a precise transition from observation to objecthood. Observation induces equivalence. Equivalence produces quotient objects. Refinement creates additional distinctions. Coarsening removes distinctions. The collection of all such policies forms a complete lattice and a thin category. Maps between state spaces transport policies contravariantly through pullback. State spaces equipped with policies assemble into a global category of identity-preserving transformations.

The resulting structure is elementary in its ingredients but foundational in interpretation. It places object formation within ordinary set theory, lattice theory, quotient construction, monoid action, and category theory. The novelty lies not in inventing unfamiliar mathematics but in recognizing that familiar mathematical structures already contain the beginnings of a formal discovery calculus.

1.1 Relation to the Published Mathematics of Concept Formation

The published paper established the broader thesis that mathematics can study not only already-defined objects but also the processes through which objects, distinctions, explanations, and theories become available. Its principal contribution was programmatic. It

proposed a discovery calculus, a geometry of explanations, a topology of error, a theory of representation-independent compression, a mathematics of emergence thresholds, a periodic system of mathematical structures, a dynamics of theories, and an algebra of observation.

The present paper develops one formal component of that program.

The published work argued that observation is not merely the passive recording of preexisting properties. Observation selects distinctions, stabilizes categories, and determines which transformations preserve identity. From that perspective, objecthood is not exhausted by membership in a preexisting ontology. An object becomes available to a theory when a family of states can be grouped under a stable criterion of equivalence and reidentified across admissible transformations.

That claim remained broad. It required a specific mathematical implementation.

The current paper supplies that implementation through equivalence relations and quotients. Every observation policy is represented by a relation of indistinguishability. Two states are equivalent when the policy assigns them the same observational identity. Their equivalence class becomes one quotient object.

This move converts a philosophical statement into a standard mathematical construction.

The published paper also proposed that concept birth should be represented as the formation of a new distinction. The present paper makes that precise. If one policy is strictly finer than another, at least one previously unified equivalence class splits into multiple classes. Relative object birth is therefore strict refinement.

Likewise, object fusion becomes coarsening. Several classes under a fine policy become one class under a coarser policy.

The earlier paper proposed that conceptual systems should possess paths, transformations, invariants, and dynamics. Here those ideas first appear categorically. Observation policies form a partially ordered family. Refinement induces canonical arrows. Composition of refinements gives categorical composition. The entire family forms a thin category.

The present work should therefore be read as a formal sequel rather than a restatement. The earlier paper supplied the conceptual horizon. This paper establishes the first theorem-bearing algebraic core.

1.2 The Need for a Formal Sequel

A broad research program is scientifically useful only if at least some of its claims can be expressed through exact definitions, propositions, and proofs.

Without formalization, phrases such as object birth, conceptual refinement, stable observation, and policy-relative identity remain suggestive but underdetermined. Different readers can interpret them differently. One reader may understand object birth psychologically, another historically, another ontologically, and another computationally. A mathematical theory requires a common structure beneath those interpretations.

The need for a formal sequel arises for several reasons.

First, the earlier paper proposed many connected ideas at once. That breadth was useful for revealing a general field, but it left open which component should be treated as foundational. Observation-induced equivalence provides a natural starting point because it is simple, rigorous, and broadly applicable.

Second, a formal sequel must separate metaphor from theorem. Terms such as geometry, topology, conservation, and emergence can be productive analogies, but they must eventually be attached to actual spaces, relations, operators, or invariants. Quotient object formation offers one case in which this attachment can be completed immediately.

Third, the broader discovery calculus requires a basic unit of conceptual organization. That unit should not be an isolated word or proposition. It should be a structure that determines which states count as identical. Equivalence relations provide exactly such a unit.

Fourth, the theory needs a principled account of change. If concepts are represented by equivalence relations, then conceptual change can be represented through refinement and coarsening. The family of possible changes already possesses order, composition, meets, joins, limits, and canonical maps.

Fifth, the theory requires compatibility with transformation. A useful object must support more than static classification. It must be possible to apply lawful transformations without ambiguity. This leads directly to the stable quotient condition.

Sixth, the theory must compare different state spaces. A policy defined on one space should be transportable along maps to another. Pullback gives the canonical mechanism.

Seventh, a formal sequel prevents premature movement into higher abstractions. Sheaves, enriched categories, higher categories, and homotopical structures may eventually become appropriate, but they should rest on a correct first-order categorical base.

The present paper therefore deliberately restricts itself. It does not attempt the entire mathematics of discovery. It isolates the smallest structure capable of carrying observation, identity, refinement, compression, and functorial transport.

That restraint is methodological. A field becomes durable when its basic objects and morphisms are clear.

1.3 Observation Before Objecthood

The ordinary order of presentation in mathematics is object first, observation second. A system is assumed to exist, and an observer then measures one of its properties.

The present framework reverses the order at the level of formal availability.

The underlying state space may exist independently of any one observation policy. However, the effective objects available to a theory depend on how that state space is partitioned by distinguishability.

Let X be a state space.

A policy E determines whether two states x and x′ should be treated as observationally identical.

If x E x′, then the policy does not preserve the distinction between x and x′.

If x is not E-equivalent to x′, then the policy distinguishes them.

The quotient X/E is therefore the observer's object space.

This does not imply that the quotient classes are arbitrary. The equivalence relation must satisfy reflexivity, symmetry, and transitivity.

Reflexivity ensures that every state is identical to itself.

Symmetry ensures that indistinguishability is mutual.

Transitivity ensures that identity is coherent across chains.

Without transitivity, one could have x equivalent to y and y equivalent to z while x remained distinct from z. Such a relation would not support stable object classes.

Observation before objecthood therefore means that object identity is induced by a policy of distinguishability. The object is not merely a point in X. It is the class of all states treated as the same under E.

Different policies produce different object worlds.

Under a fine policy, many distinctions remain visible.

Under a coarse policy, many states collapse into one effective object.

A microscopic policy may distinguish states that a macroscopic policy identifies.

A causal policy may separate states that a predictive policy groups together.

A geometric policy may ignore magnitude while preserving direction.

A dynamical policy may identify all states converging to the same attractor.

The state space is common, but objecthood varies with resolution.

This relativity does not imply that all resolutions are equally good. A policy may be unstable under the transformations relevant to the domain. It may erase distinctions later required for prediction or action. It may fail to support representative-independent operations.

Observation before objecthood is therefore only the first step. The second is stability.

1.4 Equivalence as a Policy of Identity

Every mathematical theory contains an implicit or explicit policy of identity.

In elementary arithmetic, two expressions may be treated as equal when they denote the same number.

In geometry, two figures may be treated as the same up to rigid motion, similarity, homeomorphism, or diffeomorphism.

In algebra, structures may be identified up to isomorphism.

In homotopy theory, maps or spaces may be considered equivalent under deformation.

In computation, programs may be grouped by output, semantics, or complexity.

The relevant notion of sameness depends on what the theory intends to preserve.

This paper treats an equivalence relation as a policy because it determines which substitutions are allowed without loss of identity at a chosen level.

If x E x′, then x and x′ may be substituted for one another in any operation that descends to X/E.

The phrase policy of identity emphasizes that equivalence is selective.

An equivalence relation does not declare two states identical in every possible sense. It declares them identical for a specified observational and transformational purpose.

For example, two nonzero vectors may be distinct as vectors but identical as projective directions.

Two points in the plane may be distinct as locations but identical under radial observation.

Two microstates may be distinct physically but identical under a coarse thermodynamic description.

Two datasets may be distinct as files but identical under a classifier that extracts the same label.

Policy-relative identity is therefore not a weakened form of ordinary equality. It is a more explicit account of which structure is retained and which differences are discarded.

This perspective also clarifies abstraction.

An abstraction is not merely a simpler description. It is a change in identity policy.

When several fine-grained states become one coarse object, the theory has altered what counts as the same.

The quality of the abstraction depends on whether the erased differences remain irrelevant under the transformations, predictions, and interventions that matter.

Equivalence therefore connects observation, compression, identity, and transformation.

It is the correct primitive for a mathematical theory of conceptual resolution.

1.5 Contributions and Scope

The paper makes six principal contributions.

First, it defines observation policies canonically as equivalence relations on state spaces. Observation maps remain useful as presentations, but policies with the same kernel are treated as the same resolution.

Second, it proves the stable quotient theorem. When observational equivalence is preserved by an admissible transformation monoid, every transformation descends uniquely to the quotient object space. This supplies a minimal formal criterion for stable objecthood.

Third, it organizes all observation policies on a fixed state space into a complete lattice under refinement. Meets combine distinctions. Joins generate common coarsenings. The equality relation is the finest resolution, and the universal relation is the coarsest.

Fourth, it interprets this lattice as a thin category Res(X). Refinement arrows represent lawful forgetting of distinctions. Canonical quotient maps carry fine object spaces to coarse ones.

Fifth, it proves contravariant functoriality. Every map $f : X \to Y$ pulls policies on Y back to policies on X. Observation therefore reindexes backward along representation.

Sixth, it constructs a global category whose objects are pairs (X,E), consisting of a state space and an identity policy, and whose morphisms preserve observational equivalence. Every such morphism induces a map between quotient object spaces.

The paper also develops precise interpretations of concept birth, object fusion, compression, destructive abstraction, and conceptual trajectory.

The scope remains intentionally limited.

The paper does not yet construct a non-thin discovery category containing multiple distinct refinement procedures between the same policies.

It does not yet formalize approximate, probabilistic, graded, or fuzzy equivalence.

It does not yet develop sheaf-theoretic descent for distributed observation.

It does not yet define higher morphisms or homotopies between observation procedures.

It does not claim that all scientific objects are exhausted by set-theoretic equivalence classes.

The present theory should be understood as a foundational model. It identifies one central and mathematically exact mechanism of object formation: the production of stable quotient classes under an observational identity policy.

1.6 Main Results

The principal results can be summarized in one structural sequence.

Let X be a state space.

Let Eq(X) denote the set of equivalence relations on X.

Order Eq(X) by refinement:

$E \preceq F$ exactly when E is finer than F, equivalently when E is a subset of F.

Under this order, Eq(X) is a complete lattice.

The meet of a family of policies is their intersection.

The join is the smallest equivalence relation containing their union.

The equality relation is the finest policy.

The universal relation is the coarsest policy.

The refinement order defines a thin category Res(X).

There is a unique arrow $E \to F$ exactly when $E \preceq F$.

That arrow induces a canonical surjection

$X/E \to X/F$.

The map sends each fine equivalence class to the coarse class containing it.

If a monoid G acts on X and preserves E-equivalence, then the action descends to X/E.

This is the stable quotient theorem.

If $f : X \to Y$ is a map and F is a policy on Y, then the pullback policy on X is defined by

x Pullback(f,F) x′ exactly when f(x) F f(x′).

This pullback preserves refinement and arbitrary meets.

The assignment

$X \mapsto Res(X)$

therefore defines a contravariant functor from state spaces to complete lattices of conceptual resolutions.

Finally, define a global object as a pair (X,E).

A morphism

$(X,E) \to (Y,F)$

is a function $f : X \to Y$ satisfying

x E x′ implies f(x) F f(x′).

Such a map induces a well-defined quotient map

$X/E \to Y/F$.

Together these results yield the Resolution Structure Theorem:

For every state space, its observational identity policies form a complete lattice and a thin category. Stable transformations descend to quotient object spaces. Maps of state spaces reindex observation contravariantly. Policy-equipped state spaces assemble into a global category. Concept birth, fusion, compression, and object persistence can therefore be represented within one coherent algebraic system.

The remainder of this paper develops these claims in detail. Section 2 defines state spaces, policies, and quotient objects. Section 3 proves the stable quotient theorem. Section 4 studies repeatability and persistence. Section 5 introduces refinement, birth, and fusion. Sections 6 and

7 establish the categorical and lattice structures. Section 8 treats compression and abstraction failure. Section 9 proves contravariant functoriality. Section 10 constructs the global category. Section 11 develops examples. Section 12 combines the results into the Resolution Structure Theorem and outlines the next stages of the discovery calculus.

2. State Spaces, Observation Policies, and Quotient Objects

The central objective of this paper is to identify the mathematical structure through which observational distinctions become stable objects. Rather than beginning with predefined entities, we begin with a state space and a rule that determines which states should be regarded as observationally identical. The resulting quotient space contains the effective objects available to a theory at a particular conceptual resolution.

This approach deliberately separates two questions that are often conflated.

The first concerns the underlying variability of a system.

The second concerns the distinctions that an observer, instrument, algorithm, or theory preserves.

The underlying state space may contain far more variation than is operationally significant for a given task. Observation therefore acts as a process of controlled identification rather than exhaustive discrimination.

The mathematics required for this construction is elementary. It consists of sets, equivalence relations, quotient spaces, and induced mappings. The novelty lies not in these individual ingredients but in their interpretation as the formal machinery of concept formation.

Throughout this paper, a state space represents all distinguishable configurations available before an observational policy is imposed. Observation is represented by an equivalence relation. Objects arise as equivalence classes. Refinement modifies those classes, while admissible transformations determine whether the resulting objects possess stable identity.

This sequence provides the minimal mathematical vocabulary needed for the remainder of the paper.

2.1 State Spaces and Available Variation

Let X denote a nonempty state space.

The elements of X represent the most detailed descriptions available within the chosen mathematical model. No assumption is made that these descriptions are physically

fundamental. They merely constitute the level of representation from which observational distinctions will be constructed.

The nature of X depends upon the application.

In geometry, X may consist of points of a manifold.

In dynamics, X may be a phase space.

In probability, X may be a sample space.

In machine learning, X may contain feature vectors.

In image analysis, X may contain digital images.

In biology, X may represent molecular, cellular, or physiological states.

In cognition, X may represent perceptual configurations or symbolic representations.

The theory developed here is intentionally independent of the internal structure of X.

No topology, metric, probability measure, differentiable structure, or algebraic operation is initially required.

Only the existence of distinguishable elements is assumed.

This generality is important because concept formation should not depend upon one particular mathematical setting.

The state space represents available variation before observational identification.

Nothing in the present theory requires every difference within X to remain significant.

Indeed, the purpose of observation is precisely to determine which differences should continue to matter.

2.2 Observation Maps and Observational Indistinguishability

A traditional mathematical description of observation begins with a mapping

$\pi : X \rightarrow Y,$

where Y denotes an outcome space.

Different states of X may produce identical outcomes.

Whenever

$\pi(x) = \pi(x'),$

the observation fails to distinguish x from x′.

This immediately induces a relation on X.

States are observationally indistinguishable whenever they generate identical observational outcomes.

Although observation maps provide an intuitive starting point, they contain more information than is actually required.

Two different maps may assign different numerical labels while producing exactly the same partition of X.

For example, one observer may report temperatures in degrees Celsius, another in Kelvin.

The numerical values differ, yet both induce the same distinctions.

Likewise, two classifiers may use different symbolic labels while separating the data into identical categories.

The essential mathematical content therefore lies not in the values assigned by the observation map but in the relation of observational indistinguishability that the map induces.

Consequently, observation maps should be viewed as presentations rather than as the fundamental objects of the theory.

Their kernels determine the actual conceptual resolution.

The remainder of this paper therefore treats equivalence relations as primary.

Observation maps become one convenient method of constructing them.

2.3 Equivalence Relations as Canonical Policies

An observational policy specifies when two states should be regarded as the same.

Mathematically, this is represented by an equivalence relation.

Let E be an equivalence relation on X.

Write

x E x′

when x and x′ are observationally identical under policy E.

Because E is an equivalence relation, it satisfies three familiar properties.

Reflexivity ensures that every state is identical to itself.

Symmetry ensures that observational identity is mutual.

Transitivity ensures that identity remains consistent across chains of equivalence.

These properties are not merely technical requirements.

They guarantee that observational identity defines coherent object classes.

Without transitivity, an observer could identify x with y and y with z while refusing to identify x with z.

Such inconsistency would prevent the construction of stable quotient objects.

Treating observation policies as equivalence relations has several advantages.

First, policies that induce identical partitions become mathematically identical.

Second, the theory becomes independent of arbitrary labels or measurement scales.

Third, refinement, quotient formation, and categorical structure all arise naturally from standard properties of equivalence relations.

An observational policy therefore represents a criterion of admissible substitution.

Whenever

$x \, E \, x'$,

the policy declares that replacing x by x′ preserves every distinction relevant at the chosen conceptual resolution.

Identity is thus indexed by policy rather than by absolute equality.

Different policies legitimately produce different object systems.

The mathematical problem is not to eliminate this dependence but to organize it.

2.4 Quotient Object Spaces

Given an observational policy E on X, define the quotient

X/E

to be the collection of all equivalence classes generated by E.

For each state x,

its observational object is

$[x]_E$,

the set of every state observationally identical to x.

These equivalence classes are mutually disjoint.

Every state belongs to exactly one class.

Together they partition the state space.

The quotient X/E is therefore the effective object space determined by policy E.

The distinction between X and X/E is fundamental.

Elements of X represent individual states.

Elements of X/E represent observational objects.

An object is not necessarily one state.

It is the entire family of states treated as interchangeable under E.

This interpretation has immediate consequences.

A coarse policy produces relatively few object classes.

A fine policy produces many.

The quotient therefore measures conceptual rather than physical complexity.

Changing policy changes ontology.

The underlying state space remains unchanged.

The object space changes because the equivalence relation changes.

Object formation is therefore represented mathematically as quotient formation.

This construction is entirely classical.

The novelty lies in recognizing quotient formation as the primary mathematical mechanism through which concepts become available.

2.5 Policy-Relative Identity

Equality and observational identity should not be confused.

If

$x = x'$,

the two states are literally identical.

If

$x \; E \; x'$,

they are identical only relative to policy E.

This distinction is essential.

Two vectors may be unequal while representing the same projective direction.

Two molecular configurations may differ while producing identical macroscopic observations.

Two digital images may contain different pixel values while belonging to the same recognized object class.

Two proofs may differ syntactically while expressing the same mathematical argument.

Identity therefore becomes relative to the distinctions preserved by the observational policy.

This is neither arbitrary nor subjective.

The policy specifies exactly which substitutions preserve the intended structure.

Policy-relative identity extends rather than weakens ordinary equality.

Equality is recovered as the finest possible policy.

Every state forms its own singleton class.

Conversely, the coarsest policy identifies every state with every other state.

Between these extremes lies the full spectrum of conceptual resolutions.

Scientific reasoning generally operates within this intermediate region.

Different disciplines preserve different structures because they require different notions of identity.

The present framework represents those choices explicitly.

2.6 Presentations of the Same Resolution

Observation policies should be identified by the distinctions they preserve rather than by the procedures through which they are implemented.

Suppose two observation maps

$\pi_1 : X \rightarrow Y_1$

and

$\pi_2 : X \to Y_2$

induce identical observational equivalence.

That is,

$\pi_1(x) = \pi_1(x')$

exactly when

$\pi_2(x) = \pi_2(x')$.

Although the codomains differ, both maps generate precisely the same partition of X.

From the perspective of concept formation, they represent the same observational policy.

This observation motivates an important design principle.

Policies should be identified with their induced equivalence relations rather than with their particular presentations.

Different instruments, coordinate systems, encodings, or measurement units may realize the same conceptual resolution.

The theory should identify those implementations rather than treating them as distinct objects.

This viewpoint also simplifies subsequent constructions.

Lattice operations depend only upon equivalence relations.

Categorical refinement depends only upon equivalence relations.

Stable quotient formation depends only upon equivalence relations.

The observational procedure may vary.

The induced policy remains the same.

2.7 Why Policies Should Be Identified by Their Kernels

Every observation map possesses a kernel.

The kernel consists precisely of those pairs of states producing identical observations.

Two observation maps with the same kernel determine identical conceptual resolutions.

Conversely, every equivalence relation can be realized as the kernel of an appropriate quotient map.

The quotient projection

$q : X \to X/E$

assigns every state to its equivalence class.

Its kernel is exactly E.

Thus every observational policy admits a canonical presentation.

This fact justifies taking equivalence relations as primitive.

Observation maps become secondary representations.

The quotient projection becomes canonical because it contains no unnecessary information beyond the induced partition itself.

The distinction between presentation and kernel parallels familiar ideas throughout mathematics.

Different matrices may represent the same linear transformation.

Different coordinate charts may describe the same manifold.

Different generating sets may produce the same algebraic structure.

Likewise, different observational procedures may generate the same conceptual resolution.

The kernel captures the invariant content.

Everything else belongs to the presentation.

2.8 Examples of Observational Quotienting

Several familiar mathematical constructions illustrate the general framework.

In Euclidean geometry, points in the plane may be identified according to equal distance from the origin.

Each quotient object becomes an entire circle.

The observation ignores angular position while preserving radius.

In projective geometry, nonzero vectors differing only by scalar multiplication become one object.

The quotient contains directions rather than vectors.

Magnitude has been discarded.

Orientation remains.

In graph theory, vertices may be identified through graph automorphisms.

Objects become symmetry classes rather than individual vertices.

In dynamical systems, points converging to the same attractor may be grouped into one observational object.

The quotient represents asymptotic behavior rather than initial condition.

In statistical mechanics, numerous microscopic configurations correspond to one macroscopic thermodynamic state.

The observational policy ignores microscopic variation while preserving macroscopic quantities.

In machine learning, many individual samples may receive one classification label.

The classifier defines an observational quotient.

The label represents the object.

The individual observations represent states.

These examples demonstrate that quotient object formation is neither artificial nor domain-specific.

It appears whenever observation deliberately suppresses distinctions while preserving others.

The subsequent sections ask a deeper question.

When does such a quotient support stable transformation?

Only then does the resulting conceptual resolution become suitable for systematic mathematical reasoning.

The answer to that question is provided by the stable quotient theorem developed in the next section.

3. Stable Quotients and Induced Transformation

Observation alone partitions a state space, but partitioning is not yet sufficient for stable objecthood. A useful object must support lawful transformation. If two states are treated as the same object, then every admissible transformation should send them to states that remain observationally identical. Otherwise, the action of the transformation on the quotient depends on which representative is chosen, and the quotient object fails to possess coherent dynamics.

This section develops the formal condition under which transformations descend from a state space to its quotient object space. The construction is standard in algebra, geometry, topology, and dynamics: an equivalence relation must be preserved by the transformations under consideration. The significance for concept formation is direct. A quotient class becomes a stable object only when admissible operations act consistently on the class rather than on arbitrary representatives.

The central result is the stable quotient theorem. It proves that an invariant observational policy induces a unique action on the quotient. The quotient projection then intertwines the original state-level action with the induced object-level action. This yields a rigorous distinction between lawful abstraction and destructive abstraction.

A lawful abstraction discards differences that remain irrelevant under admissible transformation.

A destructive abstraction discards differences that later become operationally significant.

Stable object formation therefore depends on the compatibility of three structures:

the state space,

the observational policy,

and the admissible transformation system.

3.1 Admissible Transformation Monoids

Let X be a state space.

A transformation of X is a function

$g : X \to X$.

The transformation may represent physical evolution, geometric motion, relabeling, computation, intervention, symmetry, or any other state change relevant to the theory.

A single transformation is usually not sufficient. Transformations can be composed, and repeated transformation should remain within the same system.

This motivates the use of a monoid.

Let G be a monoid acting on X.

The monoid G contains an identity element and supports associative composition.

For each g in G, there is a corresponding transformation

$g : X \to X$.

The identity element acts as the identity transformation on X.

For g and h in G, the product gh acts as the composite transformation obtained by first applying h and then applying g.

The monoid structure is appropriate because admissible transformations need not be invertible.

Some changes can be reversed.

Others cannot.

Rotations form a group.

Dissipative evolution generally forms only a monoid or semigroup.

Coarse-graining, compression, learning, decay, and irreversible computation are naturally noninvertible.

The theory should therefore not assume symmetry when directed transformation is sufficient.

Definition 1. Admissible transformation system

An admissible transformation system on X is a monoid G together with an action of G on X.

The term admissible indicates that these are the transformations under which the proposed object identity must remain coherent.

A quotient may be stable under one transformation system and unstable under another.

Stability is therefore always relative to G.

3.2 Invariance of Observational Identity

Let E be an observational equivalence relation on X.

The relation E is preserved by a transformation g when

$x \, E \, x'$ implies $g(x) \, E \, g(x')$.

This condition states that g respects E-equivalence.

If two states represent the same object before transformation, they must represent the same object after transformation.

Definition 2. E-compatible transformation

A transformation $g : X \to X$ is E-compatible when

x E x′ implies g(x) E g(x′)

for all x and x′ in X.

Definition 3. G-invariant observational policy

The observational policy E is G-invariant when every g in G is E-compatible.

The word invariant requires care.

It does not mean that each equivalence class remains fixed.

A transformation may send one quotient object to another.

The condition requires only that the image class depend on the original class rather than on the selected representative.

For example, a translation of the plane may move a circle centered at one point to a circle centered elsewhere. The class changes, but the transformation may still act coherently on a suitable quotient.

G-invariance is therefore invariance of identity structure, not pointwise invariance of objects.

3.3 Stable Quotient Definition

Given an equivalence relation E on X, the quotient object space is

X/E.

The canonical quotient map is

q_E : X → X/E,

defined by

q_E(x) = [x]_E.

The quotient is stable under G when every admissible transformation of X produces a well-defined transformation of X/E.

Definition 4. Stable quotient

The quotient X/E is stable under the action of G when, for every g in G, there exists a transformation

g_E : X/E → X/E

such that

g_E([x]_E) = [g(x)]_E

for all x in X.

The notation g_E denotes the quotient-level transformation induced by g.

The central issue is well-definedness.

An equivalence class contains many representatives.

The expression [g(x)]_E must be the same no matter which representative x is chosen.

This requirement is equivalent to G-invariance of E.

3.4 The Stable Quotient Theorem

Theorem 1. Stable Quotient Theorem

Let X be a state space.

Let E be an equivalence relation on X.

Let G be a monoid acting on X.

The following conditions are equivalent:

1. E is G-invariant.
2. Every g in G induces a unique transformation

g_E : X/E → X/E

defined by

g_E([x]_E) = [g(x)]_E.

3. The quotient map q_E is compatible with the G-action, meaning that for every g in G,

q_E(g(x)) = g_E(q_E(x))

for all x in X.

When these conditions hold, the induced transformations define a monoid action of G on X/E.

Proof

Assume first that E is G-invariant.

Fix g in G.

Define

g_E([x]_E) = [g(x)]_E.

To prove that g_E is well defined, suppose

[x]_E = [x']_E.

Then

x E x'.

Because E is G-invariant,

g(x) E g(x').

Therefore

[g(x)]_E = [g(x')]_E.

Hence the value of g_E does not depend on the chosen representative.

The induced map is unique because every quotient class has the form [x]_E, and its image is prescribed by [g(x)]_E.

Now let e denote the identity element of G.

Then

e_E([x]_E)
= [e(x)]_E
= [x]_E.

Thus e_E is the identity transformation on X/E.

Let g and h belong to G.

Then

g_E(h_E([x]_E))
= g_E([h(x)]_E)
= [g(h(x))]_E
= [(gh)(x)]_E
= (gh)_E([x]_E).

Therefore the induced transformations form a monoid action.

Compatibility with the quotient map follows immediately:

g_E(q_E(x))
= g_E([x]_E)
= [g(x)]_E
= q_E(g(x)).

Thus conditions 1, 2, and 3 hold.

Conversely, suppose every g induces a well-defined map g_E with

g_E([x]_E) = [g(x)]_E.

If x E x′, then

[x]_E = [x′]_E.

Applying g_E gives

[g(x)]_E = [g(x′)]_E.

Therefore

g(x) E g(x′).

Hence E is G-invariant.

The three conditions are equivalent. ■

3.5 Proof of Representative Independence

Representative independence is the essential content of the theorem.

Suppose an object is represented by the class

[x]_E.

Any state x′ satisfying

x′ E x

is another representative of the same object.

A transformation on quotient objects must assign one output class to the entire input class.

If g(x) and g(x′) belong to different E-classes, then the proposed output is ambiguous.

Using x produces one result.

Using x′ produces another.

The transformation therefore fails to act on the object itself.

It acts only on chosen representatives.

This failure shows why arbitrary partitions are not sufficient for concept formation.

A concept should permit reasoning about its instances without requiring continual access to hidden distinctions that the concept has already erased.

If the erased distinctions affect future transformation, then the quotient does not support autonomous object-level reasoning.

The stable quotient condition prevents this defect.

It ensures that every admissible operation is constant at the level of equivalence classes in the appropriate relational sense.

The object can therefore be transformed as an object.

3.6 Induced Actions on Quotient Object Spaces

When E is G-invariant, the quotient X/E inherits an action of G.

This induced action may be simpler than the original action on X.

Many distinct state-level transformations can collapse to the same quotient-level behavior.

A transformation may act nontrivially on X while becoming the identity on X/E.

For example, if X is the plane and E identifies points of equal radius from the origin, every rotation acts nontrivially on X but trivially on the quotient of radial classes.

Alternatively, a transformation may move one quotient class to another.

The quotient action then describes effective object dynamics.

This reduction is one source of explanatory power.

The state-level system may contain excessive detail.

The quotient-level system preserves only the distinctions required for object-level evolution.

Definition 5. Effective dynamics

The action of G on X/E induced by a G-invariant policy E is the effective dynamics at resolution E.

Effective dynamics need not reproduce every state-level difference.

They preserve exactly the transformations visible after quotienting.

This gives a formal account of scale-relative law.

A law at one resolution may differ from the detailed law on the underlying state space while remaining a lawful image of it.

3.7 Equivariance of the Canonical Quotient Map

The quotient map

$q_E : X \to X/E$

connects the state-level and object-level descriptions.

When E is G-invariant, q_E is equivariant.

This means that observation followed by object-level transformation gives the same result as state-level transformation followed by observation.

In words:

transform, then classify

equals

classify, then transform the class.

The commuting relation is

$q_E(g(x)) = g_E(q_E(x))$.

This relation is fundamental because it certifies that the quotient dynamics are faithful to the original transformation system at the chosen resolution.

The quotient does not invent arbitrary object-level motion.

It inherits motion from X.

Equivariance therefore provides the formal bridge between micro-level states and effective objects.

Proposition 1. Uniqueness of induced effective dynamics

If a map $h : X/E \to X/E$ satisfies

$h(q_E(x)) = q_E(g(x))$

for every x in X, then $h = g_E$.

Proof

Every element of X/E has the form q_E(x).

Therefore h and g_E agree on every quotient object. ■

The quotient-level action is not only well defined.

It is uniquely determined by the state-level action and the quotient map.

3.8 Interpretation as Lawful Object Formation

The stable quotient theorem yields a minimal formal criterion for objecthood.

An observational class becomes a lawful object when admissible transformations descend to the quotient independently of representative choice.

This criterion contains three parts.

First, the class possesses identity through equivalence.

Second, the identity persists coherently under transformation.

Third, the transformed object remains definable without recovering discarded state-level detail.

The theorem therefore distinguishes mere grouping from stable concept formation.

A grouping may be statistically convenient but dynamically unstable.

A category may predict present labels while failing under intervention.

A coarse physical variable may summarize current states while failing to support future evolution.

A concept may appear coherent until an admissible operation separates its supposedly interchangeable instances.

In each case, the quotient has erased too much.

The failure is not merely practical.

It is structural.

The operation does not descend.

Conversely, when invariance holds, the quotient becomes a legitimate level of mathematical description.

Its objects support transformations.

Its dynamics are inherited.

Its identity is representative-independent.

Its state-level and object-level descriptions are connected by an equivariant quotient map.

This justifies the interpretation of stable quotienting as lawful object formation.

The theorem also clarifies the relation between abstraction and admissibility.

An abstraction is lawful only relative to a specified transformation system.

The same quotient may be stable for one family of operations and unstable for another.

A thermodynamic macrostate may be adequate for equilibrium prediction but inadequate for reconstructing microscopic trajectories.

A classification may be stable under noise but unstable under adversarial intervention.

A geometric quotient may preserve symmetry but destroy metric information.

There is therefore no abstraction without a preservation policy.

The admissible transformations define what the concept must survive.

The observational equivalence defines what the concept is allowed to forget.

Stable object formation occurs when these two choices are compatible.

The next section develops the temporal and iterative consequences of this result. It examines repeated observation, idempotence, persistence, maintenance, and the difference between a stable object and a fixed state.

4. Repeatability, Stabilization, and Object Persistence

The stable quotient theorem establishes when an observational equivalence relation supports lawful transformation. That result alone, however, does not fully characterize mathematical objecthood. Scientific and mathematical objects are expected to persist. They must remain identifiable under repeated observation, survive admissible change, and continue to support coherent reasoning over time.

Persistence should not be confused with immobility. A planet changes position while remaining the same planet. A living organism continually exchanges matter while preserving biological identity. A software process changes internal memory while remaining the same computational

entity. Likewise, a mathematical object may undergo transformation without ceasing to be the object recognized by the observational policy.

This distinction suggests a second requirement beyond stable transformation. Observation itself should stabilize. Once a state has been assigned to an observational object, repeating the same observation policy should not continually redefine the object's identity. Repeated observation should converge to the same conceptual description unless the underlying state has changed sufficiently to cross an observational boundary.

The present section develops this idea formally. Repeated observation becomes an idempotent operation on quotient objects. Persistence becomes preservation of observational identity under admissible dynamics. Object birth and object extinction become transitions between stable observational classes rather than arbitrary changes in state. The resulting theory distinguishes transient observational artifacts from stable conceptual entities.

4.1 Repeated Observation

Scientific investigation rarely depends upon a single observation.

Measurements are repeated to verify consistency, estimate uncertainty, detect change, and establish reproducibility. The same principle applies to mathematical concept formation.

Suppose a state

x

belongs to the quotient class

[x]_E.

Applying the observational policy E once assigns x to this class.

Applying the same policy again should not produce a different object.

The first observation has already discarded every distinction ignored by E.

Nothing remains to collapse.

Observation therefore behaves differently before and after quotient formation.

Before quotienting, the policy acts on states.

After quotienting, it acts on already-formed objects.

This distinction leads naturally to an idempotent operation.

Define the object-level observation operator

Obs_E

on X/E by

Obs_E([x]_E) = [x]_E.

The operator simply recognizes the existing object.

No additional partitioning occurs.

Repeated observation therefore leaves object identity unchanged.

This property expresses conceptual stabilization.

An observational policy should create objects only once.

Subsequent application recognizes rather than reconstructs them.

4.2 Idempotent Object-Level Observation

The defining algebraic property of stabilized observation is idempotence.

Applying the same observation twice has exactly the same effect as applying it once.

Formally,

Obs_E ∘ Obs_E = Obs_E.

Proposition 1. Idempotence of stabilized observation

For every observational policy E,

the induced observation operator on X/E is idempotent.

Proof

Let

[x]_E

be any quotient object.

Applying Obs_E once gives

[x]_E.

Applying Obs_E again gives

[x]_E

once more.

Hence

Obs_E(Obs_E([x]_E))

=

Obs_E([x]_E)

=

[x]_E.

Therefore

Obs_E ∘ Obs_E = Obs_E.

■

This result appears elementary, but it carries important conceptual significance.

Object formation is not an endlessly repeated process.

Once the observational policy has identified a stable equivalence class, the same policy no longer changes the ontology.

The distinction between discovery and recognition becomes precise.

The first application produces the quotient object.

Subsequent applications merely recover it.

Idempotence therefore separates conceptual formation from conceptual use.

4.3 Stabilization of Identity

Persistence requires more than idempotent observation.

Objects must also retain their identity across admissible transformation.

Suppose the state

x

evolves under an admissible transformation

g

into

g(x).

If

g(x)

belongs to the image of

[x]_E

under the induced quotient action,

then observational identity has been preserved.

The state changes.

The object remains coherent.

Identity should therefore be understood dynamically.

An observational object is stable when every admissible evolution produces another well-defined observational object according to the induced quotient action.

This definition emphasizes that stability concerns the consistency of identity rather than the absence of change.

Many mathematical structures exhibit this behavior.

A group element changes under multiplication while remaining a group element.

A point moves continuously along a manifold while remaining a point of that manifold.

A probability distribution evolves under a Markov operator while remaining a distribution.

Likewise, quotient objects evolve through induced transformations while remaining legitimate members of the quotient space.

Persistence is therefore compatibility between identity and dynamics.

Definition 1. Persistent observational identity

An observational object possesses persistent identity under G when every admissible transformation of G acts through the induced quotient action.

Persistence is a structural property.

It depends simultaneously on the observational policy and on the transformation system.

Neither alone is sufficient.

4.4 Persistence Under Admissible Transformation

The stable quotient theorem guarantees that admissible transformations descend uniquely to the quotient.

Persistence asks what this means conceptually.

Suppose

$[x]_E$

represents one observational object.

Repeated application of transformations from G generates a sequence

$[x]_E$,

$g_1([x]_E)$,

$g_2(g_1([x]_E))$,

and so forth.

Each member of the sequence remains an observational object because every transformation acts within the quotient.

The trajectory therefore exists entirely inside the object space.

This permits reasoning directly at the conceptual level.

One no longer needs continual reference to representatives in the underlying state space.

The quotient possesses autonomous dynamics.

Such autonomy is precisely what makes abstraction scientifically valuable.

The effective object space supports prediction without reconstructing every microscopic distinction.

If the induced action did not exist, prediction at the object level would become impossible.

The observer would repeatedly need to descend to the underlying states before continuing.

Stable quotient formation avoids this dependence.

It creates a mathematically closed conceptual system.

4.5 Formation Thresholds

Not every observational grouping immediately deserves to be regarded as a stable object.

Transient equivalence classes may appear because of measurement noise, temporary coincidence, or unstable observational procedures.

The notion of stabilization suggests a formation threshold.

An observational class should be regarded as an established object only after repeated observation and admissible transformation preserve its identity.

This threshold need not be represented by a numerical constant.

Instead, it is defined structurally.

An object has crossed the formation threshold when three conditions hold.

First, the observational policy consistently assigns the same states to the same class.

Second, repeated application of the policy is idempotent.

Third, admissible transformations descend to the quotient without ambiguity.

Only then does the observational class become suitable for mathematical reasoning.

Concept formation therefore becomes a certification process rather than an instantaneous event.

Objects emerge when observational distinctions become stable enough to support transformation.

4.6 Maintenance and Extinction of Conceptual Objects

Persistence also implies the possibility of loss.

An observational object may cease to exist under a changed policy or altered transformation system.

Suppose an observational class splits after refinement.

The original object disappears, replaced by several finer objects.

Conversely, several objects may merge under coarsening.

The original distinctions are no longer maintained.

Object extinction should therefore not be interpreted as destruction of underlying states.

The states continue to exist.

Only the observational identity changes.

Likewise, object birth is not creation of new states.

It is the emergence of a new stable distinction.

This interpretation unifies birth and extinction.

Both arise from changes in conceptual resolution rather than changes in underlying ontology.

Maintenance consists of preserving observational identity across admissible transformation.

Birth introduces new stable distinctions.

Fusion removes distinctions.

Extinction occurs whenever a previously stable identity policy no longer supports an independent object.

The dynamics of concepts are therefore dynamics of equivalence classes.

4.7 Stable Identity Versus Fixed State

A common misconception identifies persistence with constancy.

Stable identity does not require an object to remain unchanged.

It requires only that the criterion of identity remain coherent.

A river changes its water continuously.

An organism replaces molecules throughout life.

A galaxy evolves over billions of years.

A mathematical proof may be rewritten while preserving logical content.

In every case, identity depends upon preserved structure rather than frozen state.

The quotient construction formalizes this distinction.

The equivalence class contains many possible representatives.

Movement among representatives need not alter object identity.

The object is defined precisely by the collection of admissible substitutions.

Consequently, persistence is fundamentally relational.

It concerns the preservation of observational equivalence rather than the immutability of representatives.

This viewpoint also explains why coarse scientific descriptions often remain stable despite enormous microscopic variation.

The observational policy deliberately ignores differences that are irrelevant to the effective level.

Those ignored differences may fluctuate continuously without altering conceptual identity.

4.8 Object Certification Through Repeatability

The combination of idempotence and transformation stability yields a practical criterion for object certification.

An observational class should be regarded as a certified object when it satisfies four conditions.

First, the policy defines a genuine equivalence relation.

Second, repeated observation is idempotent.

Third, admissible transformations descend uniquely to the quotient.

Fourth, object identity persists under the induced dynamics.

These conditions distinguish stable conceptual entities from temporary observational artifacts.

A statistical coincidence may fail repeatability.

An unstable partition may fail representative independence.

An overly coarse abstraction may fail transformation compatibility.

A noisy measurement may fail consistent classification.

Certified objects survive all of these tests.

They possess reproducible identity, lawful dynamics, and representative-independent transformation.

This notion of certification is intentionally mathematical.

It does not depend upon psychological belief or historical convention.

Certification concerns the internal coherence of the observational structure.

The object is certified because its identity remains mathematically stable under the observational policy and the admissible transformations.

The framework developed in this section completes the second stage of the theory.

Section 3 established when quotient objects support lawful transformation.

The present section establishes when those transformed objects possess stable and repeatable identity.

Together these results provide the minimal conditions for mathematical objecthood within the discovery calculus.

The following section studies how different observational policies are related through refinement and coarsening, leading to formal definitions of concept birth, object fusion, and conceptual resolution.

5. Refinement, Coarsening, Birth, and Fusion

Observation policies rarely remain fixed. Scientific instruments improve, classifications become more detailed, explanatory needs change, and previously ignored differences acquire importance. Conversely, theories often simplify by merging distinctions that no longer contribute to prediction, control, or explanation. Conceptual change therefore requires a formal account of movement between finer and coarser resolutions.

The natural relation between observation policies is refinement. A policy E is finer than a policy F when every pair of states identified by E is also identified by F. Equivalently, E preserves at least every distinction preserved by F and perhaps additional ones. Fine policies contain smaller equivalence classes. Coarse policies contain larger ones.

This ordering has immediate ontological consequences. Strict refinement splits at least one previously unified object into several objects. Coarsening merges previously distinct objects. Concept birth and object fusion are therefore not separate primitive events. They are opposite readings of the same refinement relation.

The refinement order also generates canonical maps between quotient spaces. Whenever E is finer than F, every E-object lies within one F-object. There is consequently a unique surjective map from X/E to X/F. This map formalizes the forgetting of distinctions.

The present section develops these constructions and interprets them as the first dynamics of conceptual resolution.

5.1 The Refinement Order

Let E and F be observational equivalence relations on the same state space X.

Define

$E \preceq F$

when

$x \, E \, x'$ implies $x \, F \, x'$

for all x and x' in X.

Equivalently,

$E \subseteq F$

as subsets of $X \times X$.

Under this convention, E is finer than F.

This direction may initially appear counterintuitive because the finer policy is the smaller relation. The reason is that a fine policy identifies fewer pairs. It preserves more distinctions.

The equality relation

Δ_X

is the finest possible policy because it identifies only each state with itself.

The universal relation

∇_X

is the coarsest possible policy because it identifies every state with every other state.

Thus every observational policy E satisfies

$\Delta_X \preceq E \preceq \nabla_X$.

Proposition 1. Refinement is a partial order

The relation $\preceq$ is reflexive, antisymmetric, and transitive.

Proof

Every relation E is contained in itself, so $E \preceq E$.

If $E \preceq F$ and $F \preceq E$, then $E \subseteq F$ and $F \subseteq E$. Therefore $E = F$.

If $E \preceq F$ and $F \preceq H$, then $E \subseteq F$ and $F \subseteq H$. Therefore $E \subseteq H$, so $E \preceq H$. ■

The observation policies on X therefore form an ordered space of conceptual resolutions.

5.2 Fine and Coarse Observational Policies

The refinement order can be interpreted through equivalence classes.

If $E \preceq F$, then every E-class is contained in an F-class.

Fine observation divides the state space into relatively small classes.

Coarse observation groups those classes into larger ones.

Suppose E distinguishes x from x′ while F identifies them.

Then

x is not E-equivalent to x′,

but

$x \, F \, x'$.

The difference between x and x′ is visible at resolution E and invisible at resolution F.

Resolution is therefore a measure of preserved distinction rather than simply the number of categories.

Two policies may have the same number of classes while separating different pairs of states. Such policies need not be comparable under refinement.

The refinement order is generally not linear.

Some policies preserve different and mutually incomparable distinctions.

One observer may distinguish color but ignore shape.

Another may distinguish shape but ignore color.

Neither policy is necessarily finer than the other.

This nonlinearity is important. Conceptual change is not always movement along one scale from vague to precise. It may involve changes in which dimensions count as relevant.

The complete lattice developed later will provide joint refinements and common coarsenings even when policies are incomparable.

5.3 Canonical Maps from Finer to Coarser Quotients

Let $E \preceq F$.

Each E-class lies inside exactly one F-class.

Define

$q(E,F) : X/E \to X/F$

by

$q(E,F)([x]_E) = [x]_F$.

This map sends each fine object to the coarse object containing it.

The direction is significant.

A fine quotient contains more distinctions.

A coarse quotient contains fewer.

The canonical map therefore proceeds from fine resolution to coarse resolution.

It forgets information.

Proposition 2. Well-definedness

If $E \preceq F$, then $q(E,F)$ is well defined.

Proof

Suppose

$[x]_E = [x']_E$.

Then

$x \; E \; x'$.

Because $E \preceq F$,

$x \; F \; x'$.

Therefore

$[x]_F = [x']_F$.

The value of $q(E,F)$ is consequently independent of the representative chosen. ■

The existence of this map does not require a choice of representative or any additional structure.

It follows solely from refinement.

5.4 The Canonical Surjection Theorem

Theorem 1. Canonical Surjection Theorem

Let E and F be observational policies on X.

If $E \preceq F$, then there exists a unique surjective map

$q(E,F) : X/E \to X/F$

satisfying

$q(E,F)([x]_E) = [x]_F$.

Moreover, if $E \preceq F \preceq H$, then

$q(F,H) \circ q(E,F) = q(E,H)$.

Proof

Well-definedness follows from Proposition 2.

To prove surjectivity, let $[x]_F$ be any object in X/F. The E-class $[x]_E$ exists and satisfies

$q(E,F)([x]_E) = [x]_F$.

Hence every F-object has a preimage.

Uniqueness follows because every object of X/E has the form $[x]_E$, and the required image is prescribed.

For compositionality,

$q(F,H)(q(E,F)([x]_E))$
$= q(F,H)([x]_F)$
$= [x]_H$
$= q(E,H)([x]_E)$.

Thus the canonical maps compose transitively. ■

This theorem is the quotient-level expression of conceptual coarsening.

It also anticipates the categorical structure of the next section.

5.5 Strict Refinement and Relative Object Birth

A refinement is strict when

$E \preceq F$

but

$E \neq F$.

Equivalently, E is a proper subset of F.

Strict refinement guarantees that at least one F-class splits into multiple E-classes.

Definition 1. Relative object birth

An object birth from F to E occurs when E strictly refines F.

More concretely, there exist states x and x′ such that

$x \, F \, x'$

but

x is not E-equivalent to x′.

Under F, the states belong to one object.

Under E, they belong to different objects.

The new objects are not new states.

They are newly stabilized distinctions.

This definition makes object birth relative.

A state pair may represent one object under one policy and two objects under another.

There is no contradiction because the identity criteria differ.

Corollary 1. Strict refinement produces splitting

If E strictly refines F, then at least one F-object contains at least two distinct E-objects.

Proof

Since E and F are unequal and $E \subseteq F$, there exists a pair x,x′ such that $x \, F \, x'$ but not $x \, E \, x'$.

The states lie in one F-class and in distinct E-classes. ■

Concept birth is therefore class splitting under a finer identity policy.

5.6 Coarsening and Object Fusion

The same relation can be read in the opposite direction.

If $E \preceq F$, then passage from E to F merges E-classes into larger F-classes.

Definition 2. Object fusion

An object fusion from E to F occurs when F is strictly coarser than E.

At least two distinct E-objects are mapped by q(E,F) to the same F-object.

Birth and fusion are dual descriptions of one structural change.

Moving toward finer resolution produces differentiation.

Moving toward coarser resolution produces identification.

This symmetry is useful because many scientific developments involve both operations.

A theory may refine one category while coarsening another.

For example, a new biological classification may split one species group while merging previously separated variants.

A physical theory may distinguish new particle states while unifying several forces.

Conceptual development is therefore often a mixed trajectory rather than a purely monotone refinement.

5.7 Resolution-Sensitive Ontology

The quotient space X/E represents the ontology visible at resolution E.

Changing E changes which objects exist at that level of description.

This does not require changing the underlying state space.

The ontology is resolution-sensitive because object identity depends on the chosen equivalence relation.

A fine policy may contain distinctions irrelevant at a coarser scale.

A coarse policy may support simpler laws and stronger compression.

Neither is universally superior.

The appropriate resolution depends on the transformations, interventions, and explanatory tasks that must remain valid.

This gives a precise interpretation of scale-dependent ontology.

Macroscopic objects may be legitimate even though they group many microscopic states.

Microscopic objects may be necessary when macroscopic equivalence fails under a finer intervention.

The framework therefore avoids two extremes.

It does not insist that only the finest state description is real.

It also does not treat every arbitrary coarse partition as equally valid.

A quotient ontology is legitimate when it preserves the structures required at its intended level.

Stable quotienting supplies that condition.

5.8 Conceptual Trajectories Through Refinement

A sequence of observational policies

$E_0, E_1, E_2, \ldots$

defines a conceptual trajectory.

A refinement trajectory satisfies

$E_{n+1} \preceq E_n$

for every n.

Each step introduces or preserves distinctions.

A coarsening trajectory satisfies

$E_n \preceq E_{n+1}$.

Each step removes or preserves distinctions.

A general trajectory may alternate between the two.

Such alternation is likely to be common in discovery.

Observation first differentiates cases.

Abstraction later merges them under a more general structure.

Further evidence may split the abstraction again.

Conceptual progress therefore resembles repeated refinement and coarsening rather than one-way accumulation.

For finite X, every strictly monotone trajectory terminates because only finitely many equivalence relations exist.

For infinite X, trajectories may continue indefinitely and may require limit constructions.

The complete lattice developed later supplies those limits through arbitrary meets and joins.

Conceptual trajectories also permit path comparison.

Two different sequences of refinements may arrive at the same final policy.

The thin category introduced in the next section identifies only the endpoints and order relation, not the procedural differences between paths.

That loss of path information is both a strength and a limitation.

It gives the simplest categorical structure of resolution.

It also shows why a later, richer discovery category must retain observation procedures, proofs, or costs as morphism data.

The present section establishes the ordered dynamics underlying that category.

Observation policies are related by refinement.

Refinement induces canonical quotient maps.

Strict refinement produces object birth.

Coarsening produces fusion.

The resulting resolution-sensitive ontology provides the first formal model of conceptual change within the discovery calculus.

6. The Category of Conceptual Resolutions

The refinement order developed in the previous section possesses more structure than a simple hierarchy of increasingly fine or increasingly coarse observational policies. Every refinement determines a unique and canonical relationship between two conceptual resolutions. Those relationships compose naturally, possess identity morphisms, and organize the family of observation policies into a category.

This categorical viewpoint is important because it shifts attention from individual conceptual resolutions to the relationships among them. Scientific reasoning rarely depends upon one fixed observational policy. It continually moves between different resolutions, comparing, refining, compressing, and translating conceptual descriptions. A categorical framework provides the mathematical language for these movements.

The category introduced here is intentionally minimal. There is at most one morphism between any two observational policies. Consequently, the category records only whether one policy refines another. It does not yet retain information about the procedures, costs, or evidence required to accomplish that refinement. Those richer structures belong to later developments.

The present category should therefore be regarded as the simplest algebraic skeleton of conceptual change. It captures the logical organization of observational resolutions without yet encoding the mechanisms by which one resolution is transformed into another.

6.1 Objects and Morphisms of Res(X)

Fix a state space X.

Let Eq(X) denote the collection of all observational equivalence relations on X.

Each equivalence relation represents one conceptual resolution.

These equivalence relations become the objects of the category.

Definition 1. Resolution category

The category Res(X) is defined as follows.

Its objects are the equivalence relations on X.

For two objects E and F, there exists a morphism

$E \to F$

precisely when

$E \preceq F$.

Thus every morphism represents refinement.

The direction of the morphism is significant.

It proceeds from the finer observational policy toward the coarser one.

The corresponding quotient map forgets distinctions.

The category therefore represents conceptual simplification rather than conceptual elaboration.

Moving along an arrow corresponds to replacing several finer objects with one coarser object.

Each object of Res(X) represents an entire observational ontology.

Each morphism represents lawful passage between ontologies.

6.2 Refinement Arrows

Every refinement relation determines exactly one morphism.

Unlike many familiar mathematical categories, there are no alternative arrows between the same pair of objects.

This reflects the fact that refinement is a logical relation rather than a procedural one.

If E refines F, then there is one canonical way of forgetting the distinctions preserved by E but not by F.

The associated quotient map is uniquely determined.

No additional choices are required.

The refinement arrow therefore expresses necessity rather than construction.

It records that every E-object lies inside one F-object.

The arrow contains no information about why the refinement exists.

It records only that it does.

This distinction will become important later.

A future discovery calculus may retain evidence, computation, proof, or observation history as part of the morphism itself.

The present category intentionally suppresses such information.

Only the order relation remains.

Definition 2. Refinement morphism

A refinement morphism

$E \to F$

exists if and only if

every E-equivalence class is contained within one F-equivalence class.

Equivalently,

$E \subseteq F$.

Thus refinement arrows encode inclusion of observational identity policies.

6.3 Identity and Composition

Every category requires identity morphisms and associative composition.

Both arise immediately from the refinement order.

Because every equivalence relation refines itself,

there exists an identity morphism

$E \to E$

for every object E.

Likewise,

if

$E \preceq F$

and

$F \preceq G$,

then

$E \preceq G$.

Therefore refinement arrows compose naturally.

No additional construction is required.

Composition simply records transitivity of refinement.

Theorem 1. Category theorem

For every state space X,

the observational equivalence relations on X together with refinement morphisms form a category Res(X).

Proof

Identity morphisms follow from reflexivity of refinement.

Composition follows from transitivity.

Associativity follows immediately because refinement is an ordinary binary relation.

Thus all category axioms are satisfied.

■

The categorical structure therefore requires no machinery beyond the partial order already established.

The category is the refinement order viewed through categorical language.

6.4 Thinness and Preorder Structure

The category Res(X) possesses a particularly simple form.

Between any two objects there is at most one morphism.

Such categories are known as thin categories.

Equivalently, every partially ordered set determines a thin category, and every thin category determines a preorder.

Because refinement is antisymmetric,

Res(X)

corresponds not merely to a preorder but to a genuine partial order.

Thinness has important consequences.

First, every commuting diagram commutes automatically.

There are never two distinct morphisms to compare between the same pair of objects.

Second, morphisms contain only existence information.

They do not distinguish different refinement procedures.

Third, the categorical structure is completely determined by the order relation.

This simplicity is advantageous at the foundational stage.

The category captures the logical organization of conceptual resolution while remaining mathematically transparent.

Later developments may enrich the morphisms without altering the underlying order.

6.5 Initial and Terminal Resolutions

Every state space possesses two distinguished observational policies.

The equality relation

Δ_X

distinguishes every state.

It is the finest possible resolution.

The universal relation

∇_X

identifies every state.

It is the coarsest possible resolution.

These become the boundary objects of Res(X).

Proposition 1. Initial object

The equality relation Δ_X is initial.

Proof

Since

$\Delta_X \preceq E$

for every observational policy E,

there exists a unique refinement morphism

$\Delta_X \rightarrow E$.

Therefore Δ_X is initial.

■

Proposition 2. Terminal object

The universal relation ∇_X is terminal.

Proof

Since

$E \preceq \nabla_X$

for every observational policy E,

there exists a unique refinement morphism

$E \rightarrow \nabla_X$.

Therefore ∇_X is terminal.

■

These two objects define the conceptual extremes.

The initial object preserves every distinction.

The terminal object preserves none.

Every scientific observational policy lies somewhere between complete discrimination and complete identification.

6.6 Quotient-Space Functor

The category Res(X) organizes observation policies.

Each policy also determines a quotient object space.

These quotient spaces vary coherently along refinement.

Define the assignment

Q

by

Q(E) = X/E.

If

E → F

is a refinement morphism,

define

Q(E → F)

to be the canonical quotient map

q(E,F).

This assignment respects identity and composition.

Identity refinement induces the identity quotient map.

Composed refinements induce composed quotient maps.

Theorem 2. Quotient functor

The assignment

Q

defines a covariant functor

from

$\mathrm{Res}(X)$

to the category of sets.

Proof

Identity morphisms map to identity functions on quotient spaces.

Compositionality follows from the canonical surjection theorem established in the previous section.

Therefore the functor axioms hold.

■

The quotient functor translates conceptual resolution into effective object spaces.

The category records relationships among identity policies.

The functor records the corresponding relationships among observable objects.

6.7 Conceptual Paths as Categorical Arrows

Every morphism of $\mathrm{Res}(X)$ represents one conceptual simplification.

Longer paths represent successive stages of abstraction.

For example,

suppose

$E_1 \preceq E_2 \preceq E_3$.

Then the composed morphism

$E_1 \to E_3$

represents forgetting every distinction removed during either intermediate stage.

Because the category is thin,

the direct morphism

$E_1 \to E_3$

is identical to the composite path.

The intermediate decomposition may still possess explanatory significance,

but the category itself records only the resulting refinement relation.

This observation suggests an interpretation of conceptual development.

Scientific theories often pass through many intermediate classifications before arriving at a mature conceptual organization.

The thin category remembers only the endpoint relation.

The historical pathway is invisible.

Thus Res(X) captures logical organization rather than historical process.

History belongs to a richer mathematical structure.

6.8 Limits of a Thin Discovery Category

The category Res(X) establishes the first categorical architecture of conceptual resolution.

Nevertheless, it remains intentionally limited.

Suppose two independent methods produce the same refinement.

One method may rely upon improved instrumentation.

Another may arise from theoretical insight.

A third may result from computational learning.

All three generate the same refinement arrow.

The category cannot distinguish among them.

Likewise,

proofs of refinement,

computational costs,

probabilistic confidence,

experimental evidence,

or alternative refinement algorithms

are absent.

Only the existence of refinement remains.

This limitation should not be regarded as a defect.

Thin categories frequently provide the correct starting point for more elaborate mathematical theories.

Order precedes enrichment.

The present category identifies the invariant logical structure that every richer discovery calculus should preserve.

Future work may replace refinement arrows with explicit observation procedures.

Parallel procedures could then become distinct morphisms.

Proofs comparing those procedures could become higher morphisms.

Approximate observation could introduce enriched categories.

Distributed observation could require sheaf-theoretic organization.

None of those developments alters the present result.

The first categorical structure of concept formation is already complete.

Observation policies constitute objects.

Refinement constitutes morphisms.

Composition expresses transitivity.

Identity expresses reflexivity.

Initial and terminal resolutions define conceptual boundaries.

The quotient functor connects policies with observable object spaces.

This construction establishes the minimal categorical language necessary for studying conceptual change.

The following section shows that this category possesses an even richer algebraic structure. Because arbitrary refinements admit greatest lower bounds and least upper bounds, the category is equivalent to a complete lattice of conceptual resolutions. That lattice supplies the algebraic operations through which observational distinctions can be combined, compared, and systematically reorganized.

7. Complete Lattice Structure

The category Res(X) records whether one observation policy refines another. That categorical structure is already useful, but it does not yet reveal how multiple policies can be combined. Scientific observation is often distributed across instruments, methods, scales, or theoretical perspectives. A formal theory must therefore answer two complementary questions.

What is the finest policy that preserves all distinctions made by a given family of observers?

What is the coarsest policy that contains all identifications made by that family?

These questions lead to meets and joins. The meet combines distinctions. The join combines identifications. Because arbitrary families of equivalence relations possess both operations, the category of conceptual resolutions is not merely a partially ordered set. It is a complete lattice.

This result gives the discovery calculus its first internal algebra. Observation policies can be intersected, combined, bounded, completed, and compared through universal constructions. The lattice structure also clarifies why joint observation and common abstraction are different operations. Combining distinctions usually produces a finer policy. Combining identifications produces a coarser one.

The complete lattice theorem therefore strengthens the earlier categorical result. Res(X) is not just a category whose arrows represent refinement. It is a category in which every family of conceptual resolutions possesses a greatest common refinement and a least common coarsening.

7.1 Equivalence Relations Ordered by Refinement

Fix a state space X.

Let Eq(X) denote the collection of all equivalence relations on X.

Recall that the refinement order is defined by

$E \preceq F$

exactly when

$E \subseteq F$.

Under this convention, E is finer than F because E identifies no more pairs than F.

The order is therefore opposite to the intuitive order of class size.

Fine policies contain small equivalence classes.

Coarse policies contain large equivalence classes.

The equality relation Δ_X is the least element because it is contained in every equivalence relation.

The universal relation ∇_X is the greatest element because it contains every equivalence relation.

Thus

Δ_X ≤ E ≤ ∇_X

for every E in Eq(X).

The ordered set Eq(X) is identical to the object set of Res(X), equipped with the refinement order.

The lattice operations developed below therefore become categorical limits and colimits in the associated thin category.

7.2 Meets as Combined Distinction

Suppose a family of observers uses policies

{E_i : i ∈ I}.

A state pair should remain equivalent under the joint fine resolution only if every policy identifies it.

Define the meet by

⋀{*i ∈ I*} *E_i* = ⋂{i ∈ I} E_i.

Thus

x (⋀_{i ∈ I} E_i) x′

exactly when

x E_i x′

for every i in I.

The meet preserves all distinctions made anywhere in the family.

If even one observer distinguishes x from x′, the meet distinguishes them.

For this reason, the meet is usually finer than each individual policy.

Proposition 1. Intersection preserves equivalence

The intersection of any family of equivalence relations on X is an equivalence relation.

Proof

Every E_i is reflexive, so every pair (x,x) belongs to every E_i. Therefore the intersection is reflexive.

If x is related to x′ under the intersection, then x E_i x′ for every i. By symmetry of each E_i, x′ E_i x for every i. Therefore the intersection is symmetric.

If x is related to x′ and x′ is related to x″ under the intersection, then x E_i x′ and x′ E_i x″ for every i. By transitivity of each E_i, x E_i x″ for every i. Therefore the intersection is transitive.

Hence the intersection is an equivalence relation. ■

The meet therefore always exists.

7.3 Intersections of Observation Policies

The intersection operation has a direct observational interpretation.

Suppose E records distinctions visible to observer A, while F records distinctions visible to observer B.

Then

$E \wedge F = E \cap F$

is the policy obtained when both sets of distinctions remain available.

Two states are equivalent under $E \wedge F$ only when neither observer can distinguish them.

This does not mean that the observers must use the same instrument, language, or representation. Their policies may arise from entirely different procedures. The meet operates only on the resulting equivalence relations.

For example, suppose X is a set of colored geometric figures.

Let E identify figures having the same color.

Let F identify figures having the same shape.

Then $E \wedge F$ identifies two figures only when they have both the same color and the same shape.

The joint policy is finer than either one alone.

The meet therefore acts as observational conjunction.

It retains every distinction made by any participating policy.

Theorem 1. Meet theorem

For any family $\{E_i : i \in I\}$, the relation $\bigwedge E_i$ is the greatest lower bound under refinement.

Proof

Because

$\bigcap E_i \subseteq E_j$

for every j,

$\bigwedge E_i \preceq E_j$.

Thus the intersection is a lower bound.

Now suppose H is any other lower bound. Then

$H \preceq E_j$

for every j, so

$H \subseteq E_j$

for every j.

Therefore

$H \subseteq \bigcap E_i$,

which means

$H \preceq \bigwedge E_i$.

Hence $\bigwedge E_i$ is the greatest lower bound. ■

The phrase greatest lower bound is interpreted relative to refinement. The meet is the coarsest policy among all policies finer than every E_i.

Equivalently, it is the least amount of additional distinction required to preserve all participating observations.

7.4 Joins as Generated Common Coarsenings

The corresponding operation for identifications is more subtle.

One might attempt to define the join of E and F as their union.

However, the union of equivalence relations need not be transitive.

Suppose

x E y

and

y F z,

but neither E nor F directly identifies x with z.

The union contains the first two identifications but may omit the third.

A genuine equivalence relation containing both E and F must also identify x with z by transitivity.

The join is therefore not generally the raw union.

It is the smallest equivalence relation containing the union.

Define

$\bigvee_{i \in I} E_i$

as the equivalence relation generated by

$\bigcup_{i \in I} E_i$.

This generated relation is obtained by adding every identification required by reflexivity, symmetry, and transitivity.

The join combines the identifications made by the participating policies and includes all consequences forced by those identifications.

It is therefore a common coarsening.

7.5 Equivalence Closure of Unions

The join can be described through finite chains.

Two states x and x′ are equivalent under $\bigvee E_i$ when there exists a finite sequence

$x = x_0, x_1, \ldots, x_n = x'$

such that each consecutive pair is related by at least one policy E_i.

Thus each step in the chain is licensed by one participating observer, although different steps may be licensed by different observers.

The join then identifies the endpoints through transitive closure.

This produces a significant global effect.

No single observer may have identified x with x′.

Nevertheless, the combined system may be forced to identify them because a chain of local identifications connects them.

This phenomenon is central to distributed observation.

Local equivalences can generate global equivalences that were not explicitly asserted at any one location.

Proposition 2. Existence of generated equivalence

For every relation R on X, there exists a smallest equivalence relation containing R.

Proof

Consider the family of all equivalence relations on X that contain R.

This family is nonempty because the universal relation contains every relation.

The intersection of all members of the family is itself an equivalence relation by Proposition 1.

It contains R and is contained in every equivalence relation containing R.

Therefore it is the smallest equivalence relation containing R. ■

Applying this proposition to the union of the E_i gives their join.

Theorem 2. Join theorem

For any family $\{E_i : i \in I\}$, the generated equivalence relation $\bigvee E_i$ is the least upper bound under refinement.

Proof

Every E_i is contained in the generated equivalence relation. Hence

$$E_i \preceq \bigvee E_i$$

for every i.

Thus $\bigvee E_i$ is an upper bound.

Now let H be any other upper bound. Then

$$E_i \subseteq H$$

for every i.

Therefore

$\bigcup E_i \subseteq H$.

Since H is an equivalence relation containing the union, it must also contain the smallest equivalence relation generated by that union.

Hence

$\bigvee E_i \subseteq H$,

so

$\bigvee E_i \preceq H$.

Therefore $\bigvee E_i$ is the least upper bound. ■

The join is the finest policy among all policies coarser than every E_i.

It contains every participating identification but introduces no additional identifications beyond those logically required by equivalence closure.

7.6 The Complete Lattice Theorem

A lattice is complete when every family of elements possesses both a meet and a join.

The preceding results establish this property for conceptual resolutions.

Theorem 3. Complete Lattice Theorem

For every state space X, the ordered set Eq(X) of observational equivalence relations is a complete lattice under refinement.

Arbitrary meets are intersections.

Arbitrary joins are equivalence closures of unions.

The least element is the equality relation Δ_X.

The greatest element is the universal relation ∇_X.

Proof

Theorem 1 establishes the existence of arbitrary meets.

Theorem 2 establishes the existence of arbitrary joins.

The equality relation is contained in every equivalence relation and is therefore least.

The universal relation contains every equivalence relation and is therefore greatest.

Hence Eq(X) is a complete lattice. ■

Because Res(X) is the thin category associated with Eq(X), Res(X) is a complete category in the order-theoretic sense.

Every small diagram in Res(X) possesses a limit and a colimit.

The limit corresponds to the meet of the participating policies.

The colimit corresponds to their join.

This gives the conceptual resolution system a universal algebraic organization.

7.7 Bottom and Top Resolutions

The boundary elements of the lattice deserve explicit interpretation.

The bottom resolution Δ_X is complete discrimination.

Every state is distinct from every different state.

Its quotient is naturally identified with X itself.

No compression has occurred.

The top resolution ∇_X is complete identification.

Every state belongs to one equivalence class.

Its quotient contains exactly one object, provided X is nonempty.

Maximum compression has occurred.

These extremes are mathematically necessary but scientifically unusual.

Most useful observational policies lie between them.

A theory that distinguishes everything gains maximal resolution but may lose compression, generality, and tractability.

A theory that distinguishes nothing gains maximal compression but cannot support differentiated prediction or intervention.

Concept formation therefore occurs through intermediate resolutions.

A concept is useful when it identifies enough variation to simplify reasoning while preserving enough distinction to support lawful transformation.

The lattice makes this tradeoff explicit.

Moving downward increases discrimination.

Moving upward increases identification.

Neither direction is universally progressive.

Discovery may require refinement in one region and coarsening in another.

7.8 Conceptual Resolution as an Algebra of Distinction

The complete lattice structure provides the first genuine algebra of conceptual resolution.

The meet operation combines distinctions.

The join operation combines identifications.

The bottom element represents total distinction.

The top element represents total identification.

Refinement orders policies according to preserved difference.

This algebra has several immediate consequences.

First, every family of observers possesses a canonical joint fine resolution.

Second, every family possesses a canonical common coarsening.

Third, incomparable policies can still be systematically combined.

Fourth, conceptual trajectories can possess limits.

Fifth, stabilization operators can be defined as closure or interior operators on the lattice.

Sixth, object birth and fusion can be analyzed through movement between lattice elements.

Seventh, distributed observation can be studied through local meets, joins, and generated global equivalences.

Eighth, computational searches can enumerate and optimize finite resolution lattices.

The lattice also clarifies the distinction between combining knowledge and combining ignorance.

The meet combines what observers can distinguish.

The join combines what observers are willing to identify.

These are not the same operation.

One strengthens discrimination.

The other strengthens equivalence.

In scientific practice, both are necessary.

Multiple instruments may jointly refine an object.

Multiple theories may agree to ignore different details and thereby produce a common abstraction.

The complete lattice theorem shows that both operations are canonical once policies are represented as equivalence relations.

The conceptual system therefore does not require ad hoc rules for combining observations.

Its combination laws are already supplied by lattice theory.

This result strengthens the central claim of the paper. Observation policies do not form a loose collection of classifications. They form a complete algebraic universe of possible identities on X.

The next section uses this structure to analyze quotienting as compression. It distinguishes lawful abstraction, which preserves admissible transformation, from destructive abstraction, which collapses distinctions required for coherent dynamics.

8. Compression and Destructive Abstraction

Every scientific theory suppresses information. Classical mechanics ignores molecular detail when describing planetary motion. Thermodynamics ignores microscopic particle trajectories while preserving macroscopic variables. Statistical models summarize large datasets through a comparatively small number of parameters. Machine learning replaces many individual observations with learned representations. Mathematical abstraction itself proceeds by retaining selected structures while discarding others.

From the perspective developed in this paper, every abstraction corresponds to an observational quotient. States that differ only in discarded detail are placed into the same equivalence class. The quotient therefore acts as a compression of the original state space.

Compression alone, however, does not guarantee a useful abstraction. A quotient may remove distinctions that later prove essential for prediction, transformation, or explanation. When this occurs, object-level reasoning ceases to agree with state-level reasoning. The abstraction has become destructive.

The distinction between lawful and destructive abstraction is therefore not subjective. It is determined by compatibility with admissible transformation. Stable quotient formation identifies precisely those abstractions that preserve representative-independent reasoning.

This section develops compression as a structural property of quotient formation and introduces destructive abstraction as the failure of quotient stability.

8.1 Quotient Formation as Compression

Let E be an observational policy on X.

The quotient map

$q_E : X \to X/E$

replaces every state by its observational equivalence class.

Whenever an equivalence class contains more than one state, the quotient contains fewer distinguishable objects than the original state space.

The quotient therefore compresses the representation.

Compression should not be understood merely as reduction in cardinality.

Instead, it is reduction relative to the preserved observational structure.

A quotient retains exactly those distinctions represented by its equivalence classes.

Every other distinction is intentionally discarded.

The compression performed by quotient formation is therefore semantic rather than merely numerical.

Two compressed descriptions may contain equal numbers of objects while preserving entirely different structures.

The relevant measure of compression is not simply size.

It is the identity policy embodied by the quotient.

Object formation and compression are consequently two descriptions of the same mathematical operation.

From the observational perspective, quotienting creates objects.

From the informational perspective, quotienting compresses states.

8.2 Structural Versus Cardinal Compression

Suppose two observational policies produce quotient spaces having the same number of equivalence classes.

The compressions need not be equivalent.

One policy may preserve geometric structure.

Another may preserve causal structure.

A third may preserve only statistical labels.

Cardinality alone cannot distinguish among them.

For this reason, compression should be regarded as structural.

The observational policy specifies which substitutions remain valid after quotienting.

The quotient space records the resulting ontology.

Two quotient spaces of equal size may possess completely different refinement relationships, transformation properties, and explanatory value.

Structural compression therefore depends upon preserved invariants rather than numerical reduction alone.

This viewpoint aligns naturally with mathematical abstraction.

Group theory compresses numerical calculations into symmetry classes.

Projective geometry compresses vectors into directions.

Topology compresses geometric detail while preserving continuity.

Every successful abstraction preserves some family of invariants while discarding others.

The quotient construction expresses this principle uniformly.

8.3 Compression Factors for Finite State Spaces

For finite state spaces it is useful to define a simple compression measure.

Suppose X contains N states.

Suppose the quotient X/E contains M observational objects.

The compression factor is

C(E) = N / M.

The equality policy has

C(E) = 1,

since every state remains distinct.

The universal policy has

$C(E) = N,$

since every state belongs to one observational object.

Intermediate policies produce intermediate compression factors.

Although convenient, this measure should be interpreted cautiously.

It records only numerical reduction.

It says nothing about the significance of the preserved distinctions.

A quotient reducing one million states to one thousand carefully chosen object classes may preserve every transformation relevant to a scientific theory.

Another quotient producing the same numerical reduction may destroy all predictive capability.

Compression factors therefore provide descriptive statistics rather than criteria of conceptual quality.

The decisive criterion remains transformation compatibility.

8.4 Transformation-Preserving Abstraction

An abstraction is lawful when every admissible transformation descends to the quotient.

Equivalently, observational equivalence is preserved by the transformation system.

Under this condition, object-level dynamics exist independently of representative choice.

The quotient therefore supports autonomous reasoning.

One may compute directly with quotient objects without continually recovering hidden state information.

This property distinguishes successful abstraction from arbitrary simplification.

Suppose several microscopic states correspond to one thermodynamic state.

If microscopic evolution preserves the thermodynamic equivalence relation, then thermodynamic equations constitute legitimate object-level dynamics.

Likewise, projective geometry treats scalar multiples as identical because the relevant geometric transformations preserve projective equivalence.

Transformation-preserving abstraction is therefore ubiquitous throughout mathematics.

The stable quotient theorem identifies its common algebraic structure.

Definition 1. Lawful abstraction

An observational quotient is lawful relative to G when the observational policy is invariant under every admissible transformation in G.

Lawful abstraction preserves object identity.

The quotient becomes an effective mathematical universe in its own right.

8.5 Failure of Quotient Descent

Suppose observational equivalence is not preserved.

Then two equivalent representatives may evolve into states belonging to different quotient objects.

The proposed quotient dynamics become ambiguous.

Different representatives produce different outcomes.

The transformation no longer descends.

This failure has immediate conceptual consequences.

Reasoning performed entirely within the quotient may produce incorrect conclusions.

The abstraction has discarded distinctions that later influence admissible evolution.

The quotient remains mathematically well defined as a set of equivalence classes.

What fails is its autonomy as a dynamical object space.

Representative dependence returns.

The observer must repeatedly descend to the underlying state space to determine future evolution.

The abstraction has therefore lost explanatory independence.

This phenomenon should be distinguished carefully from observational error.

The observational policy may classify states consistently.

The failure arises only when admissible transformation requires distinctions that the quotient has already removed.

The problem is structural rather than statistical.

8.6 Destructive Abstraction Theorem

The preceding discussion can be summarized as a theorem.

Theorem 1. Destructive Abstraction Theorem

Let G act on X.

Let E be an observational policy.

If E is not invariant under G, then there exists at least one admissible transformation whose action cannot be represented uniquely on X/E.

Consequently, quotient-level dynamics depend upon representative choice.

Proof

If E is not G-invariant, then there exists a transformation g in G and states x and x′ satisfying

x E x′

but

g(x) is not E-equivalent to g(x′).

Since

[x]_E = [x′]_E,

the proposed quotient action would assign two different images to one quotient object.

Therefore no well-defined induced transformation exists.

Representative dependence follows immediately.

■

The theorem provides a precise criterion for destructive abstraction.

The abstraction destroys information required for admissible transformation.

The quotient remains available as a static partition.

It fails as a stable object system.

8.7 Task-Relative Stability

Stability is always relative to a specified transformation system.

The same observational policy may be lawful for one collection of tasks and destructive for another.

For example, a biological classification suitable for ecological prediction may prove inadequate for genetic reconstruction.

A geometric abstraction preserving incidence relations may fail to preserve metric properties.

A machine learning representation effective for classification may fail for causal intervention.

There is therefore no universally optimal abstraction.

Every abstraction is indexed by the transformations it intends to support.

This observation has an important methodological consequence.

Scientific disagreement about the usefulness of an abstraction often reflects different admissible transformation systems rather than disagreement about observational data.

One investigator values predictive dynamics.

Another values intervention.

A third values symmetry.

Each chooses an observational policy appropriate for different structural goals.

The present framework accommodates all such choices.

Stability is evaluated relative to the chosen transformation monoid.

8.8 Concept Formation as Structure-Preserving Compression

The results of this section unify several themes developed earlier.

Observation induces equivalence.

Equivalence induces quotient objects.

Quotienting compresses the state space.

Compression is legitimate precisely when admissible transformation survives quotient formation.

Concept formation therefore becomes a special case of structure-preserving compression.

This interpretation clarifies the role of abstraction throughout mathematics.

Abstraction is not merely simplification.

It is controlled elimination of distinctions while preserving the structures required by the intended theory.

The quotient determines what may be forgotten.

The transformation system determines what must be remembered.

Stable objecthood exists only where these two choices are compatible.

The distinction between lawful and destructive abstraction also explains why conceptual progress is often reversible.

A coarse theory may succeed within one domain but fail under finer observation.

Refinement restores distinctions previously discarded.

Conversely, excessive detail may obscure higher-order regularities.

Coarsening reveals stable macroscopic objects invisible at microscopic resolution.

Neither refinement nor compression is inherently superior.

Both are governed by admissibility.

The observational policy should preserve exactly those distinctions required for coherent transformation and no more.

This principle provides a mathematical criterion for evaluating abstractions independently of subjective preference.

An abstraction is successful not because it is simple, elegant, or intuitive, but because it supports representative-independent dynamics on the quotient object space.

The complete lattice of conceptual resolutions developed in the previous section now acquires operational meaning.

Moving upward in the lattice increases compression.

Moving downward restores distinctions.

Stable conceptual reasoning consists of navigating this lattice while remaining within the region of transformation-preserving abstractions.

The following section extends this local theory to relationships between different state spaces. Observation policies will be transported by pullback along mappings between state spaces, establishing the contravariant functorial structure of conceptual resolution.

9. Pullback and Contravariant Observation

The preceding sections fixed a single state space X and studied the observation policies available on it. That local theory already produced stable quotients, a refinement category, and a complete lattice of conceptual resolutions. A general discovery calculus, however, must also compare different state spaces.

Scientific reasoning continually moves between representations. A physical system is mapped into measurement data. A geometric space is represented through coordinates. A biological state is mapped into a molecular signature. A dataset is transformed into a feature space. A detailed model is projected into a simpler one. In each case, an observation policy defined on the target representation induces a corresponding policy on the original state space.

The natural operation is pullback.

Suppose a map

$f : X \to Y$

represents every state of X by a state of Y. If Y carries an observational policy F, then two states of X should be considered equivalent whenever their images under f are F-equivalent.

This construction transports observation backward along representation. A policy on Y becomes a policy on X. The direction is therefore opposite to the direction of f.

This reversal is not a technical inconvenience. It expresses a basic principle:

Representations move states forward, while observations move identity policies backward.

The resulting construction is contravariantly functorial. Identity maps induce identity pullbacks. Composed state maps induce pullbacks in reverse order. Pullback also preserves arbitrary meets, meaning that combined distinctions remain combined after reindexing.

These results extend the discovery calculus from one conceptual-resolution lattice to a family of lattices indexed by state spaces.

9.1 Maps Between State Spaces

Let X and Y be state spaces.

A function

$f : X \to Y$

assigns to each state of X a state of Y.

The interpretation of f depends on context.

It may be a measurement map.

It may encode a change of representation.

It may extract features.

It may forget detail.

It may embed one state space into another.

It may represent physical evolution between differently defined spaces.

It may translate one scientific language into another.

No special assumptions are initially imposed on f.

The map need not be injective.

It need not be surjective.

It need not preserve topology, measure, algebra, or dynamics unless such structure is added later.

At the present foundational level, only the relation between state identity and observational identity matters.

Suppose Y carries an observational policy F.

The map f allows F to be interpreted on X.

Two states of X become observationally identical whenever f sends them to states identified by F.

This is the pullback of F along f.

9.2 Pullback of Observational Equivalence

Let F be an equivalence relation on Y.

Define a relation Pullback(f,F) on X by

x Pullback(f,F) x′

exactly when

f(x) F f(x′).

For compactness, this relation may also be denoted f∗F.

The notation indicates that F has been transported backward along f.

Definition 1. Pullback policy

Given $f : X \to Y$ and an observational policy F on Y, the pullback policy f^*F on X is defined by

$x \; f^*F \; x'$

if and only if

$f(x) \; F \; f(x')$.

The pullback asks whether two states of X become indistinguishable after representation in Y and observation under F.

Proposition 1. Pullback preserves equivalence

If F is an equivalence relation on Y, then f^*F is an equivalence relation on X.

Proof

For every x in X,

$f(x) \; F \; f(x)$

because F is reflexive.

Therefore

$x \; f^*F \; x$.

Thus f^*F is reflexive.

Suppose

$x \; f^*F \; x'$.

Then

$f(x) \; F \; f(x')$.

By symmetry of F,

$f(x') \; F \; f(x)$.

Hence

$x' \; f^*F \; x$.

Thus f^*F is symmetric.

Suppose

$x \; f^{*F} \; x'$

and

$x' \; f^{*F} \; x''$.

Then

$f(x) \; F \; f(x')$

and

$f(x') \; F \; f(x'')$.

By transitivity of F,

$f(x) \; F \; f(x'')$.

Therefore

$x \; f^{*F} \; x''$.

Thus f^{*F} is transitive.

Hence f^{*F} is an equivalence relation on X. ■

Every map of state spaces therefore induces a lawful reindexing of observation policies.

9.3 Monotonicity of Pullback

Pullback respects refinement.

Suppose F and H are observational policies on Y satisfying

$F \preceq H$.

Thus F is finer than H.

Every pair identified by F is also identified by H.

The same relation remains true after pullback.

Proposition 2. Pullback monotonicity

If

$F \preceq H$,

then

$f^*F \preceq f^*H$.

Proof

Suppose

$x \; f^*F \; x'$.

Then

$f(x) \; F \; f(x')$.

Because $F \preceq H$,

$f(x) \; H \; f(x')$.

Therefore

$x \; f^*H \; x'$.

Hence

$f^*F \subseteq f^*H$,

so

$f^*F \preceq f^*H$. ■

Pullback therefore defines an order-preserving map

$f^* : \mathrm{Res}(Y) \to \mathrm{Res}(X)$.

Although the assignment reverses the direction between state spaces, it preserves the refinement order within each resolution lattice.

A fine policy on Y pulls back to a fine policy relative to the distinctions visible through f.

A coarse policy pulls back to a correspondingly coarse policy.

9.4 Preservation of Arbitrary Meets

Pullback preserves joint distinction exactly.

Let $\{F_i : i \in I\}$ be a family of policies on Y.

Their meet is their intersection.

Pulling back the meet produces the same relation as meeting the individual pullbacks.

Theorem 1. Meet preservation

For every family {F_i : i ∈ I},

f∗(⋀ F_i) = ⋀ f∗F_i.

Proof

For x and x′ in X,

x f∗(⋀ F_i) x′

exactly when

f(x) (⋀ F_i) f(x′).

By definition of meet, this holds exactly when

f(x) F_i f(x′)

for every i.

That holds exactly when

x f∗F_i x′

for every i.

By definition of meet on X, this is equivalent to

x (⋀ f∗F_i) x′.

Therefore the two relations are equal. ■

This theorem has a direct observational interpretation.

If several distinctions are combined on Y and then transported to X, the result is identical to transporting each distinction separately and then combining them on X.

Joint observation is therefore stable under reindexing.

Pullback does not introduce ambiguity into the meet operation.

9.5 Identity and Composition Laws

Functoriality requires two laws.

The identity map should induce the identity operation on observation policies.

The pullback of a composite map should equal the composite of pullbacks in reverse order.

Let

$id_X : X \to X$

be the identity map.

Proposition 3. Identity law

For every observational policy E on X,

$id_X * E = E$.

Proof

For x and x′ in X,

$x \; id_X * E \; x'$

exactly when

$id_X(x) \; E \; id_X(x')$.

Since $id_X(x) = x$ and $id_X(x') = x'$, this is equivalent to

$x \; E \; x'$.

Therefore

$id_X * E = E$. ■

Now suppose

$f : X \to Y$

and

$g : Y \to Z$.

Let H be an observational policy on Z.

Proposition 4. Composition law

Pullback(g ∘ f, H) = Pullback(f, Pullback(g,H)).

Equivalently,

$(g \circ f) * H = f * (g * H)$.

Proof

For x and x′ in X,

$x\ (g \circ f)^* H\ x'$

exactly when

$g(f(x))\ H\ g(f(x'))$.

This holds exactly when

$f(x)\ g^* H\ f(x')$.

That holds exactly when

$x\ f^*(g^* H)\ x'$.

Therefore

$(g \circ f)^* H = f^*(g^* H)$. ■

The order of pullbacks is reversed relative to the order of state-space maps.

States move from X to Y to Z.

Observation policies move from Z back to Y and then back to X.

This is the defining pattern of contravariance.

9.6 The Contravariant Resolution Functor

The preceding results assemble into one categorical statement.

To every state space X, assign its complete lattice Res(X).

To every map

$f : X \to Y$,

assign the pullback map

$f^* : \mathrm{Res}(Y) \to \mathrm{Res}(X)$.

The identity and composition laws show that this assignment is functorial in the contravariant direction.

Theorem 2. Contravariant Resolution Functor

The assignment

$X \mapsto \mathrm{Res}(X)$

and

$f \mapsto f^*$

defines a contravariant functor from the category of sets to the category of complete lattices with arbitrary-meet-preserving maps.

Equivalently, it defines a covariant functor from the opposite category of sets.

Proof

Each state space X is assigned the complete lattice Res(X).

Each function $f : X \to Y$ is assigned the order-preserving, meet-preserving map

$f^* : Res(Y) \to Res(X)$.

The identity law gives

$id_X^* = id_{Res(X)}$.

The composition law gives

$(g \circ f)^* = f^* \circ g^*$.

Therefore the assignment satisfies the functor axioms in the contravariant direction.

Meet preservation was established in Theorem 1. ■

This theorem extends the discovery calculus beyond a single state space.

Conceptual resolutions now form an indexed family.

Every representation map determines how observation policies are inherited.

The framework therefore acquires a genuine global architecture.

9.7 Why Observation Reindexes Backward

Contravariance is not merely a formal reversal of arrows.

It expresses the direction in which observational identity is inherited.

Suppose f maps detailed states in X into a representation space Y.

An observer on Y distinguishes states according to policy F.

To determine which states of X appear identical through that observer, one compares their images in Y.

The policy therefore travels backward from Y to X.

The map f sends states forward.

The pullback f^* sends observational identity backward.

This pattern appears throughout mathematics and science.

A real-valued function on Y becomes a function on X by composition with f.

A subset of Y pulls back to its inverse image in X.

A topology, metric, measurable structure, or bundle may be reindexed along a map.

Likewise, an observation policy on Y induces one on X.

The conceptual meaning is especially clear in measurement.

Let X be a physical state space.

Let Y be the set of measurement outputs.

Let f assign each physical state its measurement result.

If equality is the observational policy on Y, then the pullback identifies exactly those physical states producing the same output.

The ordinary kernel of measurement is therefore a special case of pullback.

Observation by representation is inherently contravariant.

9.8 Why Pushforward Requires Additional Policy

Pullback is canonical.

Pushforward is not.

Suppose E is an observational policy on X and

$f : X \to Y$.

One may ask for an induced policy on Y.

The difficulty is that E directly constrains only points of Y that occur as images of E-related states.

It says nothing automatically about points outside the image of f.

Even within the image, different choices of extension may produce different equivalence relations on Y.

A possible construction is to generate the smallest equivalence relation on Y containing every pair

f(x), f(x′)

for which

x E x′.

Call this generated policy $f_{+}E$.

This operation may be useful, but it includes choices of interpretation.

It extends observational identity from X into Y by equivalence closure.

If f is not surjective, states outside the image remain underdetermined except for reflexive identity.

If f merges states not related by E, the resulting target-level policy may reflect properties of f rather than properties of E alone.

If additional topology, measure, dynamics, or semantics are present, several distinct pushforward policies may be reasonable.

Pushforward therefore requires an extension rule.

It is additional structure rather than part of the minimal theory.

This asymmetry is mathematically and conceptually important.

Observation inherited through a representation is automatic.

Identity projected into a larger or differently organized state space is not.

The foundational resolution construction is consequently contravariant.

Covariant constructions should be introduced only when the scientific context supplies an explicit policy for extending equivalence.

The results of this section provide the global transport law required by the discovery calculus.

Each state space carries a complete lattice of conceptual resolutions.

Every state-space map reindexes those resolutions backward.

The reindexing preserves order and arbitrary meets.

Identity and composition hold exactly.

Conceptual resolution is therefore not merely local algebra on one state space. It is a contravariantly organized system varying over the category of state spaces.

The next section combines state spaces and observation policies into a single global category. Its objects will be policy-indexed state spaces, and its morphisms will be precisely those maps that preserve observational identity. Every such morphism will induce a well-defined transformation between quotient object spaces.

10. The Global Category of Policy-Indexed State Spaces

The previous sections established two complementary structures. For every state space X there exists a complete lattice of observational policies, and every map between state spaces induces a contravariant pullback of those policies. These results organize conceptual resolution locally and explain how observation is transported between representations.

A complete theory of concept formation, however, requires one additional synthesis. State spaces and observational policies should not be treated as independent mathematical objects. Every meaningful observation is made relative to both a state space and an identity policy. The appropriate fundamental object is therefore not X alone and not E alone, but the pair (X,E).

Once these pairs become objects, the natural question concerns morphisms. A map between state spaces should qualify as a morphism only when it preserves the observational identities specified by the associated policies. Such maps induce well-defined transformations between quotient object spaces and provide the first global categorical framework for stable concept formation.

This construction unifies the local resolution lattices into a single mathematical universe. Individual state spaces become fibers indexed by observational policies, while admissible maps connect those fibers through identity-preserving transformations. Stable quotient formation, refinement, and contravariant reindexing all become components of one categorical structure.

10.1 State-Space–Policy Pairs

A state space alone specifies possible configurations.

An observational policy alone specifies a criterion of identity.

Neither by itself determines a conceptual universe.

The appropriate mathematical object consists of both simultaneously.

Definition 1. Policy-indexed state space

A policy-indexed state space is an ordered pair

(X,E),

where

X

is a state space and

E

is an observational equivalence relation on X.

The pair determines three related structures.

First, it specifies the underlying state space.

Second, it specifies the observational identity policy.

Third, it determines the quotient object space

X/E.

The quotient should therefore be regarded as derived from the pair rather than from the state space alone.

Different policies on the same state space produce different conceptual universes.

The policy-indexed pair records which universe is intended.

This viewpoint places observation at the same foundational level as the state space itself.

10.2 Identity-Preserving Maps

Suppose

(X,E)

and

(Y,F)

are policy-indexed state spaces.

A function

$f : X \to Y$

should qualify as a morphism only if observational identity is preserved.

Specifically,

whenever two states are identified by E,

their images should be identified by F.

Definition 2. Identity-preserving map

A function

$f : X \rightarrow Y$

is identity preserving when

$x \; E \; x'$

implies

$f(x) \; F \; f(x')$

for every pair of states x and x' in X.

Equivalently,

the map respects observational equivalence.

The condition depends only upon the observational policies.

No algebraic, topological, metric, or probabilistic assumptions are required.

Identity preservation expresses the minimal compatibility necessary for concept formation.

If the condition fails, two states representing one object in X may become distinct objects in Y.

Representative-independent reasoning is then lost.

The map cannot descend coherently to quotient objects.

Identity preservation therefore provides the correct notion of admissible morphism.

10.3 Composition of Policy-Compatible Transformations

Identity-preserving maps compose naturally.

Suppose

$f : (X,E) \rightarrow (Y,F)$

and

$g : (Y,F) \to (Z,H)$

are identity preserving.

If

$x \; E \; x'$,

then

$f(x) \; F \; f(x')$.

Since g preserves F,

$g(f(x)) \; H \; g(f(x'))$.

Thus

$g \circ f$

preserves observational identity.

Identity maps clearly satisfy the same condition.

Consequently, policy-compatible maps satisfy the category axioms.

Theorem 1. Composition theorem

Identity-preserving maps are closed under identity and composition.

Proof

The identity function preserves every equivalence relation because

$x \; E \; x'$

implies

$x \; E \; x'$.

Now suppose

f

and

g

are identity preserving.

The preceding argument shows that

$g \circ f$

also preserves observational identity.

Therefore identity-preserving maps satisfy the categorical axioms.

■

The proof is elementary.

Its significance is not.

Conceptual compatibility is stable under composition.

Complex conceptual translations may therefore be assembled from simpler ones.

10.4 The Global Resolution Category

The preceding definitions lead immediately to a global category.

Definition 3. Global resolution category

Define

GlobRes

to be the category whose objects are policy-indexed state spaces

(X,E),

and whose morphisms are identity-preserving maps.

Composition is ordinary function composition.

Identity morphisms are ordinary identity functions.

This category differs fundamentally from the local category

$\mathrm{Res}(X)$.

$\mathrm{Res}(X)$

contains only observational policies on one fixed state space.

GlobRes

contains every state space together with every policy defined upon it.

The local refinement lattice becomes one component of a much larger mathematical system.

Different state spaces can now be compared directly.

Different observational frameworks coexist within one categorical universe.

The theory has therefore moved from local conceptual organization to global conceptual organization.

10.5 Induced Maps Between Quotient Object Spaces

Every morphism of

GlobRes

induces a unique map between quotient object spaces.

Suppose

$f : (X,E) \to (Y,F)$

preserves observational identity.

Define

$f_Q : X/E \to Y/F$

by

$f_Q([x]_E) = [f(x)]_F.$

The definition resembles the induced transformations developed in the stable quotient theorem.

The same representative-independence argument applies.

Theorem 2. Induced quotient map theorem

Every identity-preserving morphism

$f : (X,E) \to (Y,F)$

induces a unique function

$f_Q : X/E \to Y/F.$

Proof

Suppose

$[x]_E = [x']_E.$

Then

$x \; E \; x'.$

Since

f

preserves observational identity,

f(x) F f(x′).

Therefore

[f(x)]_F = [f(x′)]_F.

The value assigned to

[x]_E

is independent of the chosen representative.

Uniqueness follows because every quotient object has the form

[x]_E.

■

This theorem establishes quotient-level functoriality.

Every admissible map between policy-indexed state spaces determines one and only one map between their observable object spaces.

10.6 Commuting Quotient Diagrams

The induced quotient map satisfies a fundamental compatibility condition.

Let

q_E : X → X/E

and

q_F : Y → Y/F

be the canonical quotient projections.

Then

f_Q(q_E(x))

=

q_F(f(x))

for every state x.

In words,

one may first classify a state and then translate the resulting object,

or first translate the state and then classify it.

The result is identical.

This commuting relation is represented symbolically by the equation

$f_Q \circ q_E = q_F \circ f$.

The quotient diagram therefore commutes.

The importance of this property extends beyond formal algebra.

It certifies that quotient-level reasoning faithfully reflects state-level reasoning.

The object space is not an independent approximation.

It is the canonical image of the underlying state space under the observational policy.

Every admissible conceptual translation respects that relationship.

Proposition 1. Uniqueness of quotient translation

If

$h : X/E \to Y/F$

satisfies

$h \circ q_E = q_F \circ f$,

then

$h = f_Q$.

Proof

Every quotient object is represented by

$q_E(x)$

for some x.

The commuting relation therefore determines the value of h on every object.

Hence

h

equals

f_Q.

■

The induced quotient translation is therefore uniquely determined by the original state-space morphism.

10.7 Recovery of the Stable Quotient Theorem

The global category contains the stable quotient theorem as a special case.

Suppose the domain and codomain coincide.

Let

(X,E)

be fixed.

Suppose

g : X → X

preserves E.

Then

g

is a morphism

(X,E) → (X,E)

inside

GlobRes.

The induced quotient map

g_Q

is precisely the quotient action established in Section 3.

Thus the stable quotient theorem becomes an internal statement about endomorphisms in the global category.

The earlier theory is therefore recovered automatically.

No additional proof is required.

This relationship clarifies the logical organization of the paper.

Section 3 developed stable quotient dynamics on one policy-indexed state space.

The present section embeds those dynamics into a category containing all policy-indexed state spaces simultaneously.

The local theory appears as one fiber inside the global one.

10.8 Policy-Indexed Ontology as a Categorical System

The global category completes the first categorical architecture of the discovery calculus.

Objects are no longer isolated state spaces.

They are state spaces equipped with explicit identity policies.

Morphisms are no longer arbitrary functions.

They are precisely those functions preserving observational identity.

Quotient object spaces arise functorially from these pairs.

Every admissible morphism induces a unique translation between quotient ontologies.

The resulting framework provides several conceptual advantages.

First, ontology becomes explicit.

Objects are always indexed by the policy that defines them.

Second, admissibility becomes structural rather than heuristic.

Maps either preserve observational identity or they do not.

Third, object translation becomes representative independent.

No ambiguity remains after quotient formation.

Fourth, local resolution lattices become components of a global mathematical universe.

Finally, the theory separates three distinct levels of organization.

The state space records available variation.

The observational policy records admissible identity.

The quotient records effective objects.

The global category organizes all three simultaneously.

This organization provides the natural setting for future developments.

Distributed observation can be represented by families of local policy-indexed state spaces.

Restriction and gluing can compare neighboring observational domains.

Higher morphisms can distinguish different conceptual translations sharing the same endpoints.

Probabilistic observation can enrich the morphism structure.

Computational discovery can search the global category for stable conceptual resolutions.

The present paper intentionally stops one step before those developments.

Its objective is to establish the minimal categorical framework within which such extensions can be defined rigorously.

The following section illustrates the theory through concrete mathematical examples. Radial observation, projective geometry, finite partition systems, dynamical basins, and classification problems will demonstrate how stable quotient formation, refinement, lattice operations, and policy-preserving morphisms arise naturally across diverse domains.

11. Examples and Model Constructions

The preceding sections developed the theory in complete generality. Observation policies were represented by equivalence relations, stable quotients were characterized through admissible transformations, conceptual resolutions were organized into complete lattices and thin categories, and policy-indexed state spaces were assembled into a global categorical framework.

The purpose of this section is to demonstrate that these constructions are not artificial. They arise naturally throughout mathematics whenever one deliberately preserves some distinctions while suppressing others. The examples are intentionally elementary. Their role is not to exhaust applications but to illustrate the fundamental mechanisms of stable quotient formation.

Each example follows the same pattern.

A state space is specified.

An observational policy is defined.

The quotient object space is identified.

Admissible transformations are examined.

The stability or instability of the resulting conceptual objects is determined.

Although the mathematical settings differ, the underlying structure remains remarkably uniform.

11.1 Radial Observation in the Plane

Let

$X = \mathbb{R}^2$.

Suppose observation preserves only distance from the origin.

Define the observational policy

E

by

(x,y) E (u,v)

exactly when

$x^2 + y^2 = u^2 + v^2$.

Each equivalence class is therefore a circle centered at the origin.

The quotient object space consists of nonnegative radii.

Every point lying on the same circle becomes one observational object.

Now consider the monoid generated by planar rotations.

Rotations preserve distance from the origin.

Consequently,

if two points belong to the same circle before rotation,

they remain on the same circle afterward.

The observational policy is invariant.

The quotient therefore supports induced dynamics.

Indeed, every rotation acts as the identity transformation on the quotient because radii remain unchanged.

The quotient removes angular information while preserving radial structure.

This example illustrates lawful abstraction.

Angular position is discarded because it is irrelevant to the admissible transformations.

The resulting object space possesses coherent dynamics.

Now replace rotations by translations.

Translations generally change distance from the origin.

Points lying on one circle may be sent to different circles.

The observational policy is no longer invariant.

The quotient action fails.

The same abstraction that was lawful for rotational symmetry becomes destructive for translational dynamics.

Stability therefore depends on the chosen transformation system.

11.2 Projective Directions from Nonzero Vectors

Let

X

be the set of all nonzero vectors in a real vector space.

Define observational equivalence by

$v \; E \; w$

exactly when

$w = \lambda v$

for some nonzero scalar λ.

Each observational object is therefore an entire one-dimensional linear subspace.

The quotient object space is projective space.

Magnitude has been discarded.

Direction has been preserved.

This quotient appears naturally throughout geometry.

Now consider the action of the general linear group.

Every invertible linear transformation sends lines through the origin to lines through the origin.

The observational policy is invariant.

Consequently,

every invertible linear transformation induces a transformation of projective space.

The quotient possesses lawful dynamics.

Projective geometry therefore arises as a stable quotient of linear geometry.

This example illustrates a general principle.

Many familiar mathematical objects originate as quotient spaces supporting induced transformation.

The present framework interprets them uniformly as concept formation through stable observational equivalence.

11.3 Finite Partitions Under Permutation Actions

Let

X

be a finite set.

Suppose an observational policy partitions X into several equivalence classes.

The quotient object space consists of those partition blocks.

Now consider the symmetric group acting on X.

Not every permutation preserves the observational policy.

A permutation preserving each equivalence class induces a quotient transformation.

A permutation mixing different classes generally does not.

Thus the admissible transformation group is the subgroup stabilizing the partition.

This example demonstrates that stability depends not upon the quotient alone but upon compatibility between the quotient and the transformation system.

The same partition may be stable under one collection of transformations and unstable under another.

The stable quotient theorem identifies precisely those admissible actions that descend.

11.4 Dynamical Basins as Quotient Objects

Suppose

X

is the phase space of a dynamical system.

Every initial condition generates a trajectory.

Assume the system possesses several attractors.

Define observational equivalence by

$x \mathrel{E} y$

when

the trajectories beginning at x and y converge to the same attractor.

Each quotient object is therefore an entire basin of attraction.

The quotient object space consists of attractor basins rather than individual trajectories.

This abstraction suppresses transient behavior while preserving long-term asymptotic structure.

If the observational policy is invariant under time evolution,

the induced quotient dynamics become trivial.

Every basin maps into itself.

The quotient has compressed detailed trajectories into stable dynamical outcomes.

This example shows that quotient formation can represent asymptotic behavior rather than instantaneous state.

Concept formation may therefore arise from future evolution rather than present configuration.

11.5 Classification Policies on Finite Datasets

Let

X

be a finite dataset.

Suppose a classifier assigns every sample one categorical label.

The observational policy identifies samples receiving identical labels.

Each quotient object is therefore one classification category.

Different classifiers generally produce different observational policies.

Some distinguish many categories.

Others identify broad classes.

Refinement corresponds to increasing classification resolution.

Coarsening corresponds to merging categories.

Now suppose a preprocessing transformation modifies the dataset.

If every pair of samples sharing one label continues to share that label after preprocessing,

the classifier is stable under that transformation.

Otherwise,

classification depends on hidden distinctions discarded by the quotient.

The abstraction becomes destructive.

This example illustrates that stable quotient formation provides a mathematical criterion for representation stability independent of any particular learning algorithm.

11.6 Observer Comparison Through Refinement

Suppose two observers examine the same state space.

Observer A employs policy

E.

Observer B employs policy

F.

If

$E \preceq F$,

then observer A distinguishes every difference visible to observer B and possibly additional ones.

Observer A therefore operates at finer conceptual resolution.

The canonical quotient map

$X/E \to X/F$

forgets exactly the distinctions visible only to A.

The refinement relation becomes a quantitative description of comparative observational power.

If neither

$E \preceq F$

nor

$F \preceq E$,

the observers preserve different distinctions.

Their policies are incomparable.

Neither observer is universally finer.

Instead,

each captures different aspects of the underlying state space.

Their joint refinement is the meet.

Their common abstraction is the join.

Observer comparison therefore becomes an algebraic operation rather than an informal judgment.

11.7 Failure Examples Under Unstable Abstraction

The preceding examples illustrated successful quotient formation.

Failure is equally informative.

Suppose

X

contains the integers.

Define observational equivalence by parity.

Even numbers form one class.

Odd numbers form another.

Now consider multiplication by two.

Parity is preserved.

The quotient remains stable.

Instead,

consider the transformation

$x \mapsto x + 1$.

Every even number becomes odd.

Every odd number becomes even.

The quotient still supports induced dynamics because parity classes are exchanged coherently.

Now modify the observational policy.

Suppose numbers are identified whenever their absolute values agree.

The classes are

{n,-n}.

Addition by one generally destroys this equivalence.

The class

{1,-1}

maps to

{2,0},

whose members are not observationally equivalent.

The induced quotient transformation fails.

The abstraction has discarded information required by admissible dynamics.

This example emphasizes that not every apparently reasonable observational policy supports stable object formation.

Compatibility with transformation remains the decisive criterion.

11.8 Computational Representation of Finite Resolution Lattices

The theory developed in this paper is directly computable for finite state spaces.

Suppose

X

contains n elements.

Every observational policy corresponds to a partition of X.

The collection of all partitions forms the complete lattice of conceptual resolutions.

Each refinement relation becomes an edge of the Hasse diagram of the lattice.

Each node represents one observational ontology.

Canonical quotient maps connect neighboring resolutions.

Conceptual trajectories become directed paths through the lattice.

Birth and fusion correspond to splitting and merging partition blocks.

The lattice therefore provides an explicit computational search space.

Algorithms may enumerate partitions, evaluate stability under chosen transformation systems, identify maximal stable compressions, compute refinement distances, locate minimal common refinements, or determine least common coarsenings.

The mathematical theory developed here therefore possesses immediate computational significance.

Concept discovery becomes the search for stable quotient structures.

Representation learning becomes exploration of admissible regions within the resolution lattice.

Compression becomes movement toward coarser stable resolutions.

Refinement becomes movement toward finer stable resolutions.

These examples demonstrate that stable quotient formation is not confined to one branch of mathematics.

Geometry, linear algebra, dynamical systems, finite combinatorics, classification theory, and computation all instantiate the same underlying construction.

Observation induces equivalence.

Equivalence induces quotient objects.

Transformation determines stability.

Refinement organizes conceptual change.

The examples also reveal that the theory is not merely descriptive.

It supplies concrete criteria for evaluating observational policies.

Some abstractions preserve lawful dynamics.

Others do not.

Some refinements reveal genuinely new stable objects.

Others merely introduce unnecessary complexity.

The mathematical distinction is supplied entirely by compatibility between observational identity and admissible transformation.

The final section synthesizes these results into one Resolution Structure Theorem and outlines the next stages of the discovery calculus.

12. Resolution Structure Theorem and Research Program

The preceding sections developed the theory in stages. Observation policies were represented by equivalence relations. Quotient spaces supplied effective objects. Transformation invariance produced stable quotient dynamics. Refinement organized object birth and fusion. The full family of policies formed a complete lattice and a thin category. Maps between state spaces pulled observation policies backward, and policy-indexed state spaces assembled into a global category.

These constructions are not independent. They form one coherent mathematical architecture. The purpose of this section is to state that architecture as a combined theorem, explain its consequences, and identify the next formal developments required for a mature discovery calculus.

The resulting structure is elementary enough to admit exact proofs but broad enough to support interpretation across geometry, dynamics, scientific observation, cognition, and artificial intelligence. Its central claim is that object formation can be represented through a disciplined sequence:

state variation,

observational equivalence,

quotient formation,

transformation stability,

refinement,

and functorial transport.

Once this sequence is made explicit, concept birth and object fusion cease to be purely metaphorical descriptions. They become operations inside a complete lattice of possible identities.

12.1 Statement of the Combined Theorem

Let X be a state space.

Let Eq(X) denote the collection of all equivalence relations on X.

Order Eq(X) by refinement:

$E \preceq F$

exactly when

$E \subseteq F$.

Thus E is finer than F whenever E identifies no more pairs than F.

Each policy E determines a quotient object space

X/E.

Whenever

$E \preceq F$,

there is a canonical surjection

$q(E,F) : X/E \to X/F$

defined by

$q(E,F)([x]_E) = [x]_F$.

Let

$f : X \to Y$

be a map of state spaces.

Every policy F on Y pulls back to a policy $f*F$ on X by

$x \; f*F \; x'$

exactly when

f(x) F f(x′).

Finally, a policy-indexed state space is a pair

(X,E),

and a morphism

f : (X,E) → (Y,F)

is a function satisfying

x E x′ implies f(x) F f(x′).

These definitions support the central theorem.

Theorem 1. Resolution Structure Theorem

For every state space X, the following statements hold.

1. The observational policies on X form a complete lattice under refinement.
2. The refinement order defines a thin category Res(X).
3. Every refinement $E \preceq F$ induces a unique canonical surjection from X/E to X/F.
4. Every monoid action on X preserving E induces a unique monoid action on X/E.
5. Every map f : X → Y induces an order-preserving pullback

$f^* : Res(Y) \to Res(X)$.

6. Pullback preserves arbitrary meets.
7. The assignment $X \mapsto Res(X)$, together with $f \mapsto f^*$, is contravariantly functorial.
8. Policy-indexed state spaces and identity-preserving maps form a global category.
9. Every morphism in that global category induces a unique map between quotient object spaces.
10. Strict refinement produces relative object birth, while strict coarsening produces object fusion.
11. Failure of transformation invariance prevents quotient dynamics from being well defined and therefore identifies destructive abstraction.

Proof

The complete lattice structure was established by showing that arbitrary intersections provide meets and equivalence closures of unions provide joins.

The thin category follows from the refinement partial order.

The canonical surjection follows from class containment under refinement.

The induced quotient action follows from the stable quotient theorem.

The pullback construction is an equivalence relation, is monotone, and preserves arbitrary intersections.

Identity and composition laws establish contravariant functoriality.

Identity-preserving functions are closed under identity and composition, producing the global category.

Representative independence gives the induced maps between quotient spaces.

Strict inclusion of equivalence relations splits or merges quotient classes.

Failure of invariance produces ambiguity in the image of a quotient class.

Therefore all stated claims hold. ■

The theorem collects the paper's principal results into one structure. It shows that observational object formation possesses lattice, categorical, dynamical, and functorial organization simultaneously.

12.2 Complete Lattice of Observation Policies

The complete lattice is the static algebra of conceptual resolution.

Every policy occupies a definite location between total discrimination and total identification.

The finest policy is the equality relation.

The coarsest policy is the universal relation.

Every family of policies has a meet.

The meet preserves every distinction made by any member of the family.

Every family also has a join.

The join contains every identification made by the family and every additional identification forced by equivalence closure.

This structure is important because scientific observation is rarely singular. Different instruments and theories distinguish different aspects of a system. The lattice provides canonical operations for combining them.

The meet answers:

What is the coarsest policy that still preserves all participating distinctions?

The join answers:

What is the finest policy that still contains all participating identifications?

These universal constructions replace informal comparison with exact algebra.

The lattice also supplies limits of monotone conceptual trajectories.

A descending refinement chain has an intersection representing its limiting resolution.

An ascending coarsening chain has a generated join representing its limiting abstraction.

Conceptual change therefore need not be modeled only through isolated steps. It can possess convergence, stabilization, and fixed points.

12.3 Thin Categorical Structure

The category Res(X) is the dynamic order skeleton of the lattice.

Its objects are observation policies.

Its arrows point from finer policies to coarser policies.

Each arrow represents the lawful forgetting of distinctions.

Because the category is thin, the existence of an arrow is equivalent to refinement. There is no separate procedural information.

This thinness yields clarity.

Composition is simply transitivity.

Identity is reflexivity.

Every diagram commutes automatically.

Initial and terminal objects are given by total discrimination and total identification.

Meets and joins become categorical limits and colimits.

The category therefore expresses conceptual organization with minimal machinery.

Its limitation is equally clear.

It cannot distinguish two different methods that yield the same refinement. It does not retain evidence, cost, confidence, algorithm, historical pathway, or proof object.

This limitation marks the boundary between the present theory and a richer discovery calculus.

Res(X) records the order of possible resolutions.

It does not yet record the process of discovering them.

12.4 Functorial Variation of Quotient Object Spaces

The quotient-space assignment

$E \mapsto X/E$

is functorial on Res(X).

Whenever E refines F, the finer quotient maps canonically to the coarser quotient.

Thus object spaces do not vary arbitrarily as policies change.

They vary according to a lawful family of surjections.

This yields a precise hierarchy of ontologies.

Fine resolutions contain more objects.

Coarse resolutions contain fewer.

The canonical maps record exactly which fine objects fuse under coarsening.

Functoriality also ensures that multi-stage abstraction is coherent.

If E refines F and F refines H, then passing from X/E to X/H directly gives the same result as passing first through X/F.

No ambiguity arises from the chosen route.

This is the first formal expression of compositional concept change.

The theory therefore contains not only objects and policies but a structured relationship between changes in policy and changes in ontology.

12.5 Contravariant Reindexing Across State Spaces

The resolution construction varies contravariantly across state spaces.

A map

$f : X \to Y$

sends states forward.

An observation policy on Y moves backward to X by pullback.

This direction is unavoidable.

To determine whether two states of X appear identical under a policy on Y, one compares their images.

Observation therefore follows inverse image rather than direct image.

The contravariant functor

$X \mapsto \mathrm{Res}(X)$

organizes conceptual resolution across representations.

It shows that observation is not attached to isolated state spaces. Policies can be inherited along measurement maps, coordinate maps, encodings, and feature maps.

The preservation of arbitrary meets is especially important.

Joint distinctions on Y remain joint distinctions after transport to X.

Thus combined observational power is stable under reindexing.

By contrast, joins need not be preserved in general. A map may collapse paths of identification in ways that alter equivalence closure. This asymmetry indicates that pullback preserves conjunctive distinction more directly than generated common coarsening.

The contravariant structure therefore contains information about which conceptual operations survive changes of representation.

12.6 Concept Birth, Fusion, and Abstraction Failure

The lattice gives exact meanings to three central processes.

Concept birth occurs through strict refinement.

A previously unified quotient class splits into multiple classes.

No new underlying state is required.

A new object appears because the identity policy has become more discriminating.

Object fusion occurs through strict coarsening.

Previously distinct quotient classes map to one larger class.

Again, no state is destroyed.

A distinction is removed.

Abstraction failure occurs when an equivalence relation is not preserved by an admissible transformation.

The quotient still exists as a static set, but it cannot support representative-independent dynamics.

The abstraction has removed information required by the transformation system.

These three processes provide the first genuine operations of the discovery calculus.

Birth adds distinction.

Fusion removes distinction.

Failure exposes an illegitimate identification.

Together they describe the elementary dynamics of ontology.

12.7 Relation to the Broader Discovery Calculus

The published mathematics of concept formation proposed a broad research program involving discovery, explanation, error, compression, emergence, observation, and theory dynamics.

The present paper formalizes only one part of that program, but it provides a base for the rest.

Explanation can be studied relative to paths through resolution lattices.

A change in explanation may correspond to refinement, coarsening, or movement between incomparable policies.

Error can be represented as selection of an unstable or task-inappropriate quotient.

A misconception may identify states that admissible transformations later separate.

Compression is already represented by coarsening.

The remaining problem is to define task-sensitive measures of structural loss.

Emergence can be represented as stabilization of a quotient whose classes support autonomous effective dynamics.

Observation is represented by policy selection and pullback.

Theory change can be modeled as a trajectory through a family of resolution lattices, possibly accompanied by changes in the underlying state spaces themselves.

The present theory therefore supplies the ontology layer of the broader calculus.

It explains how possible states become effective objects.

Later theories must explain how policies are selected, evaluated, learned, revised, and coordinated.

12.8 Next Steps: Descent, Enriched Morphisms, and Computational Discovery

The immediate research program should proceed in a disciplined order.

The first next step is local observation and descent.

Suppose a state space is covered by observational domains. Each domain carries a local equivalence relation. One must determine when these local policies glue into a global policy and when transitive closure creates new global identifications.

This problem naturally leads to presheaves, sheaves, and descent conditions.

The second step is enrichment of morphisms.

The thin category remembers only that one policy refines another. A richer category should distinguish different procedures producing the same refinement.

Morphisms may contain observation algorithms, proofs, data, costs, or intervention protocols.

Parallel morphisms would then become meaningful.

The third step is approximate observation.

Real observation is rarely exact. Policies may be probabilistic, graded, noisy, or context-dependent.

Equivalence relations may need to be replaced or supplemented by tolerances, kernels, enriched relations, or probabilistic channels.

The fourth step is discovery dynamics.

Closure operators can model stabilized coarsening.

Interior operators can model stabilized refinement.

Fixed points can represent conceptually stable policies.

Iterative operators can generate trajectories through resolution lattices.

The fifth step is computational search.

For finite state spaces, the full partition lattice can be enumerated.

Algorithms can test stability under transformation monoids, calculate compression, identify maximal stable quotients, and trace refinement pathways.

The first computational implementation should not be presented as a general artificial intelligence system. It should be a transparent finite laboratory for testing the mathematics.

The sixth step is policy selection.

The present theory describes possible policies but does not choose among them.

A complete discovery calculus requires admissibility criteria involving prediction, intervention, compression, robustness, and explanatory value.

The seventh step is higher structure.

Only after morphisms retain procedural information should higher morphisms be introduced. They may compare refinement procedures, certify equivalence of discovery paths, or represent homotopies between observational transformations.

These developments should remain anchored to the exact base established here.

The principal achievement of the present paper is not speculative reach but mathematical closure.

Observation policies have been defined.

Their refinement order has been proved complete.

Their arrows have been organized categorically.

Their quotient spaces vary functorially.

Their transport across state spaces is contravariant.

Their compatible pairs form a global category.

Their births, fusions, and failures have exact meanings.

The theory therefore establishes the first rigorous algebraic structure of the discovery calculus.

Mathematics usually studies transformations of already-formed objects. The Resolution Structure Theorem shows how the formation of those objects can itself be organized mathematically.

A state space contains possibilities.

An observational policy determines identity.

A quotient produces objects.

Stable transformations certify those objects.

Refinement creates new distinctions.

Coarsening compresses them.

Functoriality transports them across representations.

The formal birth of objects is therefore not one isolated event. It is a lawful position within a complete categorical system of possible conceptual resolutions.

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